MASS AND WIDTH OF $\sigma(750)$ SCALAR MESON FROM MEASUREMENTS OF $\pi N \to \pi^- \pi^+ N$ ON POLARIZED TARGETS

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Abstract.

The measurements of reactions $\pi^- p_\uparrow \to \pi^-\pi^+ n$ at 17.2 GeV/c and $\pi^+ n_\uparrow \to \pi^+\pi^- p$ at 5.98 and 11.85 GeV/c made at CERN with polarized targets provide a model-independent and solution-independent evidence for a narrow scalar state $\sigma(750)$. The original χ^2 minimization method and the recent Monte Carlo method for amplitude analysis of data at 17.2 GeV/c are in excellent agreement. Both methods find that the mass distribution of the measured amplitude $|\overline{S}|^2\Sigma$ with recoil transversity "up" resonates near 750 MeV while the amplitude $|S|^2\Sigma$ with recoil transversity "down" is large and nonresonating. The amplitude $|S|^2\Sigma$ contributes as a strong background to S-wave intensity $I_S = (|S|^2 + |\overline{S}|^2)\Sigma$ and distorts the determinations of σ resonance parameters from I_S. To avoid this problem we perform a series of Breit-Wigner fits directly to the measured distribution $|\overline{S}|^2\Sigma$. The inclusion of various backgrounds causes the width of $\sigma(750)$ to become very narrow. Our best fit with t-averaged coherent background yields $m_\sigma = 753 \pm 19$ MeV and $\Gamma_\sigma = 108 \pm 53$ MeV. These values are in excellent agreement with Ellis-Lanik theorem for the width of scalar gluonium. The gluonium interpretation of $\sigma(750)$ is also supported by the absence of $\sigma(750)$ in reactions $\gamma\gamma \to \pi\pi$. We also show how data on polarized target invalidate essential assumptions of past determinations of $\pi\pi$ phase shifts which explains the absence of $\sigma(750)$ in the conventional phase shift δ_0^0. We examine the interference of $\sigma(750)$ with $f_0(980)$ and find it has only a very small effect on the determination of $\sigma(750)$ mass and width. The data on amplitude $|\overline{S}|^2\Sigma$ in the mass range of 1120–1520 MeV show existence of a scalar resonance $f_0(1300)$ with a mass of 1280 $\pm$ 12 MeV and a width of 192 $\pm$ 26 MeV. We point out that the study of production processes on the level of spin amplitudes measured in experiments with polarized targets may reveal new hadron structures and open new physics beyond the standard QCD quark model.

I. Introduction

Science is an ongoing interplay between our ideas and our experiences in the real Universe. We cannot expect to make a progress in our understanding unless we can show that what we thought we knew is in some sense incomplete or even wrong.[1] Amplitude analyses of hadronic reactions provide a clear confirmation of this logic of scientific discovery.

In 1972, L. van Rossum and his group at Saclay measured recoil nucleon polarization in $\pi^{\pm}p \to \pi^{\pm}p$ reactions[2] at CERN-PS. Their results closed the set of complete measurements of $\pi N \to \pi N$ reactions at 6 GeV/c and allowed them to make the first model independent determination of hadronic amplitudes directly from scattering data.[3] Failures of Regge models to correctly predict polarization and other spin observables were traced to the wrong structure of their amplitudes. Some Regge models were revised and constrained to reproduce the experimental πN amplitudes at 6 GeV/c.

In 1978 the Saclay group reported[4] the first amplitude analysis of KN and $\overline{K}N$ charge exchange reactions, also at 6 GeV/c. The structure of A_2-exchange amplitudes was found different from all revised Regge models. A more difficult amplitude analysis of $pp \to pp$ at 6 GeV/c was made possible by polarized proton beam at Argonne ZGS.[5] The results reported[6] in 1985 by A. Yokosawa and his group at Argonne confirmed our lack of understanding of hadronic reactions at the level of amplitudes. So far, no new revisions of Regge models were even attempted. When the first measurement of a complete set of observables in elastic scattering of protons on ^{12}C carbon nucleus was reported[7] in 1981, it invalidated the standard nonrelativistic analysis in favour of relativistic approach to nuclear physics.[8]

The work by Saclay, Argonne and other groups firmly established the need for experimental knowledge of hadronic amplitudes. Efficient acquisition of this knowledge in two-body reactions has been hampered by the difficulty of measuring the recoil nucleon polarization. Dispersion relations were used in an effort to obtain hadronic amplitudes from incomplete data.[9] However, the situation is different in pion production reactions $\pi N \to \pi\pi N$ and $KN \to K\pi N$. In 1978, Lutz and Rybicki showed[10] that measurements of pion production in meson-nucleon scattering on transversely polarized target yield in a single experiment enough observables that almost complete and model independent amplitude analysis can be performed.

Amplitude analyses of pion production reaction such as $\pi N \to \pi\pi N$ or

$NN \to \pi NN$ are important for two special reasons. First, these reactions provide information about unnatural exchange amplitudes which are not accessible in two-body reactions. Second, such amplitude analyses enable us to study resonance production on the level of spin-dependent amplitudes rather than spin-averaged cross-section $d^2\sigma/dmdt$.

The high statistics measurement of $\pi^-p \to \pi^-\pi^+n$ at 17.2 GeV/c at CERN-PS on unpolarized target[11] was later repeated with a transversely polarized proton target at the same energy[12−−16] Model independent amplitude analyses were performed for various intervals[12−−15] of dimeson mass of small momentum transfers $-t = 0.005$–0.2 (GeV/c)^2 and over a large interval[16] of momentum transfer $-t = 0.2 - 1.0$ (GeV/c).^{2}

Additional information was provided by the first measurement of $\pi^+n \to \pi^+\pi^-p$ and $K^+n \to K^+\pi^-p$ reactions[17,18] on polarized deuteron target at 5.98 and 11.85 GeV/c, also done at CERN-PS. The data allowed to study the t-evolution of mass dependence of moduli of amplitudes.[19] Detailed amplitude analyses[20,21] determined the mass dependence of amplitudes at larger momentum transfers $-t = 0.2$–0.4 (GeV/c).^{2}

Amplitude analyses of $\pi^-p \to \pi^-\pi^+n$ and $\pi^+n \to \pi^+\pi^-p$ reactions were recently repeated[22,23] with special attention paid to error propagation and selection of physical solutions. This work was motivated by the emerging evidence from previous analyses for a new scalar resonance $\sigma(750)$.

All amplitude analyses[12−23] of pion production on polarized targets found a clear evidence for large and nontrivial unnatural A_1-exchange amplitudes in the dipion mass range from 400 to 1800 MeV. This experimental finding is very important since previously the A_1 exchange amplitudes were assumed absent. In particular, all determinations of $\pi\pi$ phase shifts from unpolarized data on $\pi^-p \to \pi^-\pi^+n$ are based on the assumption of vanishing A_1 exchange amplitudes.[24−−31] Without this assumption the determination of $\pi\pi$ phase shifts cannot even proceed.[23] The existence of large and nontrivial A_1 exchange amplitudes in $\pi N \to \pi^+\pi^-N$ reactions casts a serious doubt about the reliability of the conventional $\pi\pi$ phase shifts. The assumption of absence of A_1 exchange amplitudes means that pion production in $\pi N \to \pi^+\pi^-N$ does not depend on nucleon spin. What the measurements of $\pi N \to \pi^+\pi^-N$ on polarized targets found is that pion production depends strongly on nucleon spin. The dynamics of pion production is therefore not as simple as was

assumed in the past determinations of $\pi\pi$ phase shifts.

The existence of A_1 exchange is also crucial for our understanding of spin structure of the nucleon. The measurements of the cross-section asymmetry using longitudinally polarized lepton beams on a longitudinally polarized nucleon targets determine nucleon spin dependent structure function $g_1(x, Q^2)$. The measurements of g_1 on proton and neutron targets at CERN and SLAC has fascinating implications for the internal structure of the nucleon. These analyses of nucleon spin structure depend on the behaviour of g_1 for $x \to 0$ which is controlled by A_1 exchange (see eq. (4.2.23) of Ref. 32).

Another important finding of measurements of $\pi^- p \to \pi^- \pi^+ n$ and $\pi^+ n \to \pi^+ \pi^- p$ reactions on polarized targets is the evidence for a narrow scalar state $I = 0$ $0^{++}(750)$.

The first published evidence for this state in $\pi^- p \to \pi^- \pi^+ n$ is presented in Fig. 2 of Ref. 14 from 1979. The CERN-Munich data on S-wave intensity I_S show a clear bump around 750 or 800 MeV (depending on a solution). The CERN-Munich group refers to this bump as a resonance $\epsilon(800)$ but makes no Breit-Wigner fits. In 1979, Donohue and Leroyer published[33] an analysis of the CERN-Munich data on polarized target and made the first claim that the data show existence of a narrow resonance which they called $\epsilon(750)$.

Our Saclay group measured $\pi^+ n \to \pi^+ \pi^- p$ on polarized deuteron target at 5.98 and 11.85 GeV/c at CERN–PS. The measured S-wave intensity I_S showed a narrow resonant structure at 750 MeV at both energies (see Fig. 10 of Ref. 20). To verify our results on I_S in $\pi^+ n \to \pi^+ \pi^- p$ reaction, we analysed the CERN-Munich data at 17.2 GeV/c using our computer programs. All four solutions for S-wave intensity I_S at all 3 energies showed a narrow resonant structure around 750 MeV. We reported these results in 1992 in Ref. 22 and made the claim for existence of a narrow scalar state $\sigma(750)$. However, later a numerical error was found in the final step of calculation of S-wave intensity I_S at 17.2 GeV/c. When corrected, only 2 solutions for I_S at 17.2 GeV/s show a clear narrow structure around 750 MeV while the other two solutions have a broader behaviour. Thus there appeared some difference between the results at low momentum transfer data at 17.2 GeV/c and higher momentum transfer data at 5.98 and 11.85 GeV/c.

In Ref. 22 we presented analytical solutions of our amplitude analyses which included the unphysical solutions in many (m, t) bins. To deal with the unphysical

solutions and to improve our error analysis we used a Monte Carlo method of amplitude analysis in Ref. 23. A clear signal for a narrow $\sigma(750)$ emerged from this improved analysis. This evidence for a narrow $\sigma(750)$ state was recently confirmed in a new measurement of $\pi^- p \to \pi^- \pi^+ n$ on polarized target at 1.78 GeV/c made at ITEP accelerator in Moscow and reported[34] in 1995.

To describe our new results, compare them to the CERN-Munich analyses, and to outline the program of this paper, we first need to introduce our notation and discuss the two accepted methods for amplitude analysis of data on polarized targets.

For masses below 1 GeV the dimeson system is produced predominantly in spin states $J = 0$ (S-wave) and $J = 1$ (P-wave). The experiments yield 15 spin-density-matrix (SDM) elements describing the dimeson angular distribution. These observables and the cross-section $d^2\sigma/dmdt \equiv \Sigma$ can be expressed in terms of two S-wave and six P-wave nucleon transversity amplitudes.[10,20] In our notation, the two S-wave amplitudes are S and $\overline{S}$. The six P-wave amplitudes are $L, \overline{L}, U, \overline{U}$ and $N, \overline{N}$. The amplitudes $\overline{A} = \overline{S}, \overline{L}, \overline{U}, \overline{N}$ and $A = S, L, U, N$ correspond to recoil nucleon transversity "up" and "down" relative to the scattering plane. The amplitude analysis works with normalized amplitudes using a normalization

$$|S|^2 + |\overline{S}|^2 + |L|^2 + |\overline{L}|^2 + |U|^2 + |\overline{U}|^2 + |N|^2 + |\overline{N}|^2 = 1 \tag{1.1}$$

The unnormalized amplitudes then are $|A|^2\Sigma$ and $|\overline{A}|^2\Sigma$. We also define partial-wave intensity

$$I_A = (|A|^2 + |\overline{A}|^2)\Sigma \tag{1.2}$$

where $A = S, L, U, N$. There are two solutions for the moduli $|A|^2$ and independent two solutions for the moduli $|\overline{A}|^2$. Hence, there are 4 independent solutions for the partial wave intensities I_A which we label $I_A(i,j), i,j = 1,2$ with indices i and j referring to the two solutions for $|A|^2$ and $|\overline{A}|^2$, respectively.

Amplitude analysis expresses analytically[10,20] the eight normalized moduli and the six cosines of relative phases of nucleon transversity amplitudes in terms of measured SDM elements. There are two similar solutions in each (m,t) bin. However, in many (m,t) bins the solutions are unphysical: either a cosine has magnitude larger than one or the two solutions for moduli are complex conjugate with a small imaginary part. Unphysical solutions also complicate the error analysis.

The occurrence of unphysical solutions is a common difficulty in all amplitude analyses. Two methods are used to find physical solutions and determine their errors. They are (a) χ^2 minimization method and (b) Monte Carlo method.

In the χ^2 method one minimizes a function

$$\chi^2 = \sum_{i=1}^{M} [\frac{\mathrm{Obs}_i(\mathrm{meas}) - \mathrm{Obs}_i(\mathrm{calc})}{\Delta_i}]^2 \tag{1.3}$$

where Obs_i (meas) are the experimentally measured quantities, Δ_i are their experimental errors and Obs_i (calc) are corresponding expressions in terms of the amplitudes (moduli and cosines of relative phases). The analytical solutions for the moduli and cosines serve as initial values. This χ^2 method was used in all CERN-Munich analyses[12−16] of $\pi^- p \to \pi^-\pi^+ p$ at 17.2 GeV/c. Since the two analytical solutions (initial values) are very close, the χ^2 method leads to a unique solution in many (m, t) bins. A particular exception is the mass range below 900 MeV. More recently the χ^2 method was used in direct reconstruction of amplitudes of pp elastic amplitudes from 0.8 to 2.7 GeV using polarized data obtained at SATURN II at Saclay.[35]

The basic idea of Monte Carlo method is to vary randomly the input SDM elements within their experimental errors and perform amplitude analysis for each new set of the input SDM elements. The resulting moduli and cosines of relative angles are retained only when all of them have physical values in both analytical solutions. Unphysical solutions are rejected. The distributions of accepted moduli and cosines define the range of their physical values and their average value in each (m, t) bin. The Monte Carlo amplitude analysis of Ref. 23 is based on 30,000 random variations of the input SDM elements. The Monte Carlo method was first used in 1977 in an amplitude analysis[36] of pp elastic scattering at 6 GeV/c and later in an amplitude analysis[37] of reactions $\pi^- p \to K^+K^- n$ and $\pi^- p \to K^0_S K^0_S n$ at 63 GeV/c. In his review paper,[38] F. James advocates the use of the Monte Carlo method as perhaps the only way to calculate the errors in the case of nonlinear functions which produce non-Gaussian distributions. The method has the added advantage that it can separate the physical and unphysical solutions and that it can retain the identity of the two analytical solutions.

The results for the two solutions for the unnormalized moduli of S-wave amplitudes $|\overline{S}|^2\Sigma$ and $|S|^2\Sigma$ obtained by Monte Carlo amplitude analysis of CERN-Munich data at 17.2 GeV/c are shown in Fig. 1. We find that both solutions for the

amplitude $|\overline{S}|^2\Sigma$ resonate around 750 MeV while both solutions for the amplitude $|S|^2\Sigma$ show non-resonant behaviour and increase with dipion mass m.

The results for $|\overline{S}|^2\Sigma$ and $|S|^2\Sigma$ obtained by χ^2 minimization method using the same CERN-Munich data at 17.2 GeV are shown in Fig. 2. We again find that both solutions for the amplitude $|\overline{S}|^2\Sigma$ resonate around 750 MeV and that both solutions for the amplitude $|S|^2\Sigma$ show non-resonant behaviour and increase with dipion mass m. The comparison of Fig. 1 and Fig. 2 shows that Monte Carlo method and χ^2 minimization method are also in excellent numerical agreement. However, the amplitudes obtained by Monte Carlo method show a considerably smoother behaviour which, as we shall see later, gives much lower χ^2 values in Breit-Wigner fits.

The unnormalized moduli $|\overline{S}|^2\Sigma$ and $|S|^2\Sigma$ in Fig. 1 and 2 were calculated using $\Sigma = d^2\sigma/dmdt$ from Fig. 12 of Ref. 11.

At this point we note that the Fig. 2 is based on Fig. 10 of Ref. 13 and Fig. VI–21 of Ref. 12 (to resolve error bars). The authors of these papers present only normalized moduli $|\overline{S}|$ and $|S|$ and consequently did not see the resonant behaviour of unnormalized amplitude $|\overline{S}|^2\Sigma$. The resonant behaviour of amplitude $|\overline{S}|^2\Sigma$ at 750 MeV went also unobserved in the subsequent analysis in Ref. 14 which was using polarized data in 40 MeV bins in the mass range from 600 to 1800 MeV. It is possible to reconstruct the amplitudes $|\overline{S}|^2\Sigma$ and $|S|^2\Sigma$ from the information given in Ref. 14. As we shall see in Section V (Fig. 8) both solutions for $|\overline{S}|^2\Sigma$ resonate below 900 MeV while both solutions for $|S|^2\Sigma$ are nonresonating, in agreement with Figures 1 and 2. It is interesting to note that the evidence for a narrow resonance $\sigma(750)$ was hidden in the very first analyses of CERN-Munich data (Ref. 12, 13 and 14) and was recognized by the present author some 16 years later in connection with the present work.

The aim of the present work is a more reliable determination of mass and width of $\sigma(750)$ resonance from model independent amplitude analyses of CERN-Munich data on $\pi^- p \to \pi^- \pi^+ n$ on polarized target at 17.2 GeV/c. There are three important issues that we address in the process.

The first issue in the question which mass distribution should be used for Breit-Wigner fits to determine the resonance parameters of $\sigma(750)$ state. The previous CERN-Munich analyses[14–16] fitted a Breit-Wigner formula to partial wave intensities, and we followed the same procedure in Ref. 23. However, the S-wave intensity

at lower momentum transfers at 17.2 GeV/c shows a clear resonant structure only in solutions $I_S(1,1)$ and $I_S(2,1)$ while the solutions $I_S(1,2)$ and $I_S(2,2)$ lack sufficient decreases of I_S above 800 MeV to indicate a narrow resonance. This behaviour in $I_S = (|\overline{S}|^2 + |S|^2)\Sigma$ is caused by the large and nonresonating amplitude $|S|^2\Sigma$. The amplitude $|S|^2\Sigma$ thus behaves as a large and nonresonating background to the resonating amplitude $|\overline{S}|^2\Sigma$ and this distorts the determination of resonance parameters of $\sigma(750)$ from Breit-Wigner fits to I_S. To avoid this problem, it is necessary to perform Breit-Wigner fits directly to the resonant mass distributions $|\overline{S}|^2\Sigma$. Both solutions to $|\overline{S}|^2\Sigma$ resonate and the evidence for $\sigma(750)$ is thus entirely solution independent.

The second issue is which resonance shape formula is to be used in Breit-Wigner fits to $|\overline{S}|^2\Sigma$. The previous analyses[11,14––16,23] used the Pišút-Roos shape formula which multiplies the standard Breit-Wigner formula (with a phase space) by an additional mass dependent factor $F = (2J+1)(m/q)^2$. In their analysis of $\pi N \to \pi\pi N$ reaction amplitudes,[39] Pišút and Roos assumed absence of A_1 exchange amplitudes and assumed that the mass dependence of pion production amplitudes is given by $\pi\pi$ scattering amplitudes. The partial wave expansion of $\pi\pi$ amplitudes then directly leads to the additional factor F. However, because of the existence of large and nontrivial A_1 exchange amplitudes in $\pi N \to \pi^+\pi^- N$ reactions and because there is no proof that the mass dependence of $\pi N \to \pi\pi N$ production amplitudes is really described by the mass dependence of partial waves in $\pi\pi$ scattering, it is useful to perform Breit-Wigner fits to $|\overline{S}|^2\Sigma$ mass distribution using the standard[40] phenomenological shape formula in which the Pišút-Roos factor is absent (i.e. $F = 1$) to see if there are differences in the determination of resonance parameters of $\sigma(750)$. The comparison of fits using both shape formulas finds only small differences.

The third issue is the question of background in the resonant mass distribution $|\overline{S}|^2\Sigma$. Nonresonating background comes e.g. from the isospin $I = 2$ contribution to $|\overline{S}|^2\Sigma$. While it is difficult to exactly parametrize the unknown background, we estimated the background contribution using 3 different approaches. In each case we find that inclusion of background leads to a significant reduction of the width of $\sigma(750)$ to somewhere around 100 MeV. Background is obviously important for the width determination of $\sigma(750)$ and thus to our understanding of the constituent structure of the $\sigma(750)$ state. We also examine the interference of $\sigma(750)$ with

$f_0(980)$ and find it has only a small effect on the mass and width of $\sigma(750)$.

The paper is organized as follows. In Section II we review the basic formalism. In Section III we derive the Pišút-Roos shape formula and describe the phenomenological shape formula for Breit-Wigner fits. In Section IV we present our fits to the measured resonating amplitude $|\overline{S}|^2\Sigma$ and describe our approaches to inclusion of coherent background. In Section V we study the interference of $\sigma(750)$ with $f_0(750)$ and also perform fits in the broad mass range 600–1520 MeV which show evidence for a scalar resonance $f_0(1300)$. In Section VI we present our fits to S-wave intensity I_S in $\pi^- p \to \pi^-\pi^+ n$ at 17.2 GeV/c and in $\pi^+ n \to \pi^+\pi^- p$ at 5.98 and 11.85 GeV/c. In Section VII we review the assumptions of past determinations of $\pi\pi$ phase shifts and show how they are invalidated by the data on polarized targets. This explains the absence of narrow $\sigma(750)$ state in conventional phase shift δ_0^0. In Section VIII we answer critical questions concerning the evidence for a narrow $\sigma(750)$. In Section IX we propose to identify the $\sigma(750)$ state with lowest mass gluonium $0^{++}(gg)$ and discuss theoretical and experimental support for this interpretation of $\sigma(750)$. In Section X we comment on the significance of studying hadron production on the level of spin amplituds and suggest that such studies may reveal new physics beyond the conventional QCD quark model of hadrons. The paper closes with a summary in Section XI.

II. Basic Formalism

A. Phase space and amplitudes.

Various aspects of phase space, kinematics and amplitudes in pion production in $\pi N \to \pi\pi N$ reactions are described in several books.[41–44] In our discussion we will follow often the book by Pilkuhn.[41]

Consider reaction $a+b \to 1+2+3$ such as $\pi^- p \to \pi^-\pi^+ n$ with four-momentum conservation

$$P = p_a + p_b = p_1 + p_2 + p_3 = p_d + p_3 \tag{2.1}$$

where $p_d = p_1 + p_2$ is the dipion momentum. The spin averaged cross-section is given by

$$d\sigma = \frac{1}{\text{Flux}(s)} \frac{1}{(2s_a+1)(2s_b+1)} \sum_{\lambda_n,\lambda_p} |M_{\lambda_n,0\lambda_p}|^2 d\text{Lips}_3 \tag{2.2}$$

where the flux

$$\text{Flux}(s) = 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2} = 4MP_{\pi lab} \tag{2.3}$$

with $m_a = \mu$ mass of pion and $m_b = M$ mass of proton. The λ_p and λ_n are the proton and neutron helicities. The dipion state does not have a definite spin and helicity. The Lorentz invariant phase space is defined as

$$d\text{Lips}_3 = (2\pi)^4 \delta^4(P - p_1 - p_2 - p_3) \prod_{i=1}^{3} \frac{d^3 p_i}{(2\pi)^3 (2E_i)} \tag{2.4}$$

We will work with the usual kinematic variables of c.m. energy squared s, momentum transfer t, dipion mass m and angles θ, ϕ describing the angular distribution of π^- in the $\pi^-\pi^+$ rest frame. Hence

$$\begin{aligned} s &= (p_a + p_b)^2 = (p_1 + p_2 + p_3)^2 \\ t &= (p_a - p_d)^2 = (p_a - p_1 - p_2)^2 \\ m^2 &= p_d^2 = (p_1 + p_2)^2 \end{aligned} \tag{2.5}$$

Following the procedure described on pp. 18–19 of Ref. 40 and using $dm^2 = 2mdm$, we can write

$$d\text{Lips}_3(P, p_1, p_2, p_3) = q(m^2) G(s) dm dt d\Omega$$

where q is the pion momentum in the c.m.s. of the dipion system

$$q(m^2) = \sqrt{0.25m^2 - \mu^2} = \frac{m}{2}\sqrt{1 - (\frac{2\mu}{m})^2} \tag{2.6a}$$

and the energy dependent part of phase space

$$G(s) = \frac{1}{(2\pi)^4} \frac{1}{8\sqrt{\lambda(s,\mu^2,M^2)}} \tag{2.6b}$$

where

$$\lambda(s,\mu^2,M^2) = [s-(\mu+M)^2][s-(\mu-M)^2]$$

Hence

$$\frac{d\sigma}{dmdtd\Omega} = \frac{K(s)}{4\pi} q \sum_{\lambda_p,\lambda_n} |M_{\lambda_n,0\lambda_p}(s,t,m,\theta,\phi)|^2 \tag{2.7}$$

where

$$K(s) = \frac{2\pi G(s)}{\mathrm{Flux}(s)}$$

To obtain dipion states of definite spin J and helicity λ, we expand

$$M_{\lambda_n,0\lambda_p} = \sum_{J=0}^{\infty} \sum_{\lambda=-J}^{+J} \sqrt{(2J+1)} M^J_{\lambda\lambda_n,0\lambda_p}(s,t,m) d^J_{\lambda 0}(\theta) e^{i\lambda\phi} \tag{2.8}$$

We now integrate $|M_{\lambda_n,0\lambda_p}|^2$ over $d\Omega$. Using orthogonality relations for the d-functions and spherical harmonics, we obtain for the reaction cross-section

$$\frac{d^2\sigma}{dmdt} = q(m^2)K(s) \sum_{J=0}^{\infty} \sum_{\lambda,\lambda_n,\lambda_p} |M^J_{\lambda\lambda_n,0\lambda_p}(s,t,m)|^2 \tag{2.9}$$

We now define

$$H^J_{\lambda\lambda_n,0\lambda_p} = \sqrt{q(m^2)}\sqrt{K(s)} M^J_{\lambda\lambda_n,0\lambda_p} \tag{2.10}$$

We will also consider only the $J=0$ (S-wave) and $J=1$ (P-wave) contributions for dipion masses m below 1000 MeV. With a notation

$$\Sigma = d^2\sigma/dmdt \tag{2.11}$$

we now define normalized helicity amplitudes with definite t-channel naturality

$$\begin{aligned} H^0_{0+,0+} &= S_0\sqrt{\Sigma}, \quad H^0_{0+,0-} = S_1\sqrt{\Sigma} \\ H^1_{0+,0+} &= L_0\sqrt{\Sigma}, \quad H^1_{0+,0-} = L_1\sqrt{\Sigma} \\ H^1_{\pm1+,0+} &= \frac{1}{\sqrt{2}}(N_0 \pm U_0)\sqrt{\Sigma} \\ H^1_{\pm1+,0-} &= \frac{1}{\sqrt{2}}(N_1 \pm U_1)\sqrt{\Sigma} \end{aligned} \tag{2.12}$$

In (2.12) $n = |\lambda_p - \lambda_n| = 0, 1$ is the nucleon helicity flip. At large s, the unnatural helicity nonflip amplitudes S_0, L_0, U_0 and the unnatural helicity flip amplitudes S_1, L_1, U_1 exchange π and A_1 quantum numbers in the t-channel, respectively. Both natural exchange amplitudes N_0 and N_1 exchange A_2 at large s.

The amplitude analysis of data on polarized targets is performed using normalized recoil nucleon transversity amplitudes defined as

$$\begin{aligned}
S &= \frac{1}{\sqrt{2}}(S_0 + iS_1) & \overline{S} &= \frac{1}{\sqrt{2}}(S_0 - iS_1) \\
L &= \frac{1}{\sqrt{2}}(L_0 + iL_1) & \overline{L} &= \frac{1}{\sqrt{2}}(L_0 - iL_1) \\
U &= \frac{1}{\sqrt{2}}(U_0 + iU_1) & \overline{U} &= \frac{1}{\sqrt{2}}(U_0 - iU_1) \\
N &= \frac{1}{\sqrt{2}}(N_0 - iN_1) & \overline{N} &= \frac{1}{\sqrt{2}}(N_0 + iN_1)
\end{aligned} \tag{2.13}$$

The amplitudes S, L, U, N and $\overline{S}, \overline{L}, \overline{U}, \overline{N}$ correspond to recoil nucleon transversity "down" and "up", respectively.[18,20] The "up" direction is the direction of normal to the scattering plane defined according to Basel convention by $\vec{p}_\pi \times \vec{p}_{\pi\pi}$ where $\vec{p}_\pi$ and $\vec{p}_{\pi\pi}$ are the incident pion and dimeson momenta in the target nucleon rest frame.

The normalized amplitudes satisfy conditions

$$\sum_{n=0}^{1} |S_n|^2 + |L_n|^2 + |U_n|^2 + |N_n|^2 = 1$$

$$|S|^2 + |\overline{S}|^2 + |L|^2 + |\overline{L}|^2 + |U|^2 + |\overline{U}|^2 + |N|^2 + |\overline{N}|^2 = 1 \tag{2.14}$$

We now define spin-averaged partial wave intensities for amplitudes $A = S, L, U, N$

$$I_A = (|A_0|^2 + |A_1|^2)\Sigma = (|A|^2 + |\overline{A}|^2)\Sigma \tag{2.15}$$

Obviously

$$\Sigma = \frac{d^2\sigma}{dmdt} = I_S + I_L + I_U + I_N \tag{2.16}$$

¿From the point of view of Breit-Wigner fits to various mass distributions in $\pi^- p \to \pi^- \pi^+ n$ ($\Sigma, I_A, |\overline{S}|^2\Sigma$ etc.) the important point is that the part of the phase space which depends on the dipion mass m is simply the c.m. pion momentum q given by (2.6a).

B. Spin observables and amplitude analysis.

For invariant masses below 1000 MeV, the dipion system in reactions $\pi N \to \pi^+\pi^- N$ is produced predominantly in spin states $J = 0$ (S-wave) and $J = 1$ (P-wave). The experiments on transversely polarized targets then yield 15 spin-density-matrix (SDM) elements describing the dipion angular distribution. The measured SDM elements are[17,18]

$$\rho_{ss} + \rho_{00} + 2\rho_{11}, \rho_{00} - \rho_{11}, \rho_{1-1}, \tag{2.17a}$$

$$\mathrm{Re}\rho_{10}, \mathrm{Re}\rho_{1s}, \mathrm{Re}\rho_{0s},$$

$$\rho^y_{ss} + \rho^y_{00} + 2\rho^y_{11}, \rho^y_{00} - \rho^y_{11}, \rho^y_{1-1}, \tag{2.17b}$$

$$\mathrm{Re}\rho^y_{10}, \mathrm{Re}\rho^y_{1s}, \mathrm{Re}\rho^y_{0s},$$

$$\mathrm{Im}\rho^x_{1-1}, \mathrm{Im}\rho^x_{10}, \mathrm{Im}\rho^x_{1s}. \tag{2.17c}$$

The SDM elements (2.17a) are also measured in experiments on unpolarized targets. The observables (2.17b) and (2.17c) are determined by the transverse component of target polarization perpendicular and parallel to the scattering plane $\pi N \to (\pi^+\pi^-)N$, respectively. The SDM elements (2.17) depend on s, t, and m. There are two linear relations among the matrix elements in (2.17):

$$\begin{aligned} \rho_{ss} + \rho_{00} + 2\rho_{11} &= 1, \\ \rho^y_{ss} + \rho^y_{00} + 2\rho^y_{11} &= A, \end{aligned} \tag{2.18}$$

where A is the polarized target asymmetry.

The data analysis is carried out in the t-channel helicity frame for the $\pi^+\pi^-$ dimeson system. The helicities of the initial and final nucleons are always defined in the s-channel helicity frame. Using the notation of (2.17) and (2.18), the first group of equations represents the sum of SDM elements (2.17a) and (2.17b):

$$\begin{aligned} a_1 &= \frac{1}{2}[1 + A] = |S|^2 + |L|^2 + |U|^2 + |\overline{N}|^2, \\ a_2 &= [(\rho_{00} - \rho_{11}) + (\rho^y_{00} - \rho^y_{11})] = 2|L|^2 - |U|^2 - |\overline{N}|^2, \\ a_3 &= [\rho_{1-1} + \rho^\nu_{1-1}] = |\overline{N}|^2 - |U|^2, \end{aligned} \tag{2.19a}$$

$$\begin{aligned} a_4 &= \frac{1}{\sqrt{2}}[\mathrm{Re}\rho_{10} + \mathrm{Re}\rho^y_{10}] = |U||L|\cos(\gamma_{LU}), \\ a_5 &= \frac{1}{\sqrt{2}}[\mathrm{Re}\rho_{1s} + \mathrm{Re}\rho^y_{1s}] = |U||S|\cos(\gamma_{SU}), \\ a_6 &= \frac{1}{2}[\mathrm{Re}\rho_{0s} + \mathrm{Re}\rho^y_{0s}] = |L||S|\cos(\gamma_{SL}). \end{aligned} \tag{2.19b}$$

Similar equations relate the difference of SDM elements to amplitudes of opposite transversity. The second group of observables is defined as

$$\begin{aligned}
\overline{a}_1 &= \frac{1}{2}[1 - A] = |\overline{S}|^2 + |\overline{L}|^2 + |\overline{U}|^2 + |N|^2, \\
\overline{a}_2 &= [(\rho_{00} - \rho_{11}) - (\rho^y_{00} - \rho^y_{11})] = 2|\overline{L}|^2 - |\overline{U}|^2 - |N|^2, \\
\overline{a}_3 &= [\rho_{1-1} - \rho^y_{1-1}] = |N|^2 - |\overline{U}|^2, \qquad (2.20a) \\
\overline{a}_4 &= \frac{1}{\sqrt{2}}[\mathrm{Re}\rho_{10} - \mathrm{Re}\rho^y_{10}] = |\overline{U}||\overline{L}|\cos(\overline{\gamma}_{LU}), \\
\overline{a}_5 &= \frac{1}{\sqrt{2}}[\mathrm{Re}\rho_{1s} - \mathrm{Re}\rho^y_{1s}] = |\overline{U}||\overline{S}|\cos(\overline{\gamma}_{SU}), \\
\overline{a}_6 &= \frac{1}{2}[\mathrm{Re}\rho_{0s} - \mathrm{Re}\rho^y_{0s}] = |\overline{L}||\overline{S}|\cos(\overline{\gamma}_{SL}), \qquad (2.20b)
\end{aligned}$$

In Eqs. (2.19b) and (2.20b) we have introduced explicitly the cosines of relative phases between the nucleon transversity amplitudes.

The SDM elements (2.17c) form the third group of observables,[10,20] which is not used in the present amplitude analysis.

The first group (2.19) involves four moduli $|S|^2$, $|L|^2$, $|U|^2$ and $|\overline{N}|^2$ and three cosines of relative phases $\cos(\gamma_{SL})$, $\cos(\gamma_{SU})$, and $\cos(\gamma_{LU})$. The second group (2.20) involves the same amplitudes, but with opposite nucleon transversity. Analytical solution for these amplitudes in terms of observables was derived in Ref. 10 and 20. For the first group one obtains a cubic equation for $|L|^2 \equiv x$:

$$ax^3 + bx^2 + cx + d = 0, \qquad (2.21)$$

with coefficients a, b, c, d expressed in terms of observables $a_i, i = 1, 2, \ldots, 6$. The remaining moduli and the cosines are given by the expressions

$$\begin{aligned}
|S|^2 &= (a_1 + a_2) - 3|L|^2, \\
|U|^2 &= |L|^2 - \frac{1}{2}(a_2 + a_3), \\
|\overline{N}|^2 &= |L|^2 - \frac{1}{2}(a_2 - a_3), \\
\cos(\gamma_{LU}) &= \frac{a_4}{|L||U|}, \\
\cos(\gamma_{SU}) &= \frac{a_5}{|S||U|}, \qquad \cos(\gamma_{SL}) = \frac{a_6}{|S||L|}. \qquad (2.22)
\end{aligned}$$

The solution for the second group (2.20) is similar.

The analytical solutions of the cubic equation (2.21) are given in Table I of Ref. 20. One solution of (2.21) is always negative and it is rejected. The other two solutions are generally positive and close. However, in a number of (m, t) bins we get unphysical values for some cosines and in some cases also negative moduli of amplitudes. In some (m, t) bins the mean values of input SDM elements yield complex solutions for $|L|^2$ or $|\overline{L}|^2$ or both (with positive real parts). To filter out the unwanted unphysical solutions and to determine the errors on the amplitudes and their average values, one can use either the χ^2 minimization method or the Monte Carlo method. The results of amplitude analyses of $\pi^- p \to \pi^- \pi^+ n$ at 17.2 GeV/c for dipion masses in the range 600–900 MeV are given in Ref. 13 for the χ^2 minimization method and in Ref. 23 for the Monte Carlo method. The two methods are in excellent agreement. In Ref. 23 we also present the Monte Carlo amplitude analysis of the reaction $\pi^+ n \to \pi^+ \pi^- p$ at 5.98 and 11.85 GeV/c using the Saclay data[17] at larger momentum transfers $-t = 0.2 - 0.4$ (GeV/c)^2 and dipion masses in the range 360–1040 MeV.

III. Resonance shape formulas.

A. Pišút-Roos shape formula.

Before we review the Pišút-Roos derivation of their resonance shape formula, we first recall some properties of partial waves in elastic scattering of scalar particles. The T-matrix amplitude of isospin I has partial wave expansion

$$T^I(s, \cos\theta) = 8\pi \sum_{L=0}^{\infty} (2L+1) T_L^I(s) P_L(\cos\theta) \tag{3.1}$$

The unitarity in elastic scattering then requires (see Ref. 40, pp. 38–40) that T_L^I has a form

$$T_L^I(s) = \frac{\sqrt{s}}{q} \sin\delta_L^I e^{i\delta_L^I} = \frac{\sqrt{s}}{q} \frac{1}{\text{cotg}\delta_L^I - i} \tag{3.2}$$

where q is the pion c.m.s. momentum and δ_L^I is the corresponding phase shift. Notice that the factor $\sqrt{s}/q$ is induced by the unitarity alone. At a resonance m_R, the relativistic Breit-Wigner formula for T_L^I then reads

$$T_L^I = \frac{\sqrt{s}}{q} \frac{-m_R \Gamma(s)}{(s - m_R^2) + i m_R \Gamma(s)} \tag{3.3}$$

where $\Gamma(s)$ is an energy dependent width.

Let us now return to pion production process $\pi^- p \to \pi^-\pi^+ n$ and to amplitudes $M^J_{\lambda\lambda_n,0\lambda_p}(s,t,m)$ defined in (2.8). In their analysis,[39] Pišút and Roos assumed that the following amplitudes vanish for all J:

$$M^J_{0+,0+} = 0 \tag{3.4a}$$

$$M^J_{\pm1+,0+} = M^J_{\pm1+,0-} = 0 \tag{3.4b}$$

The conditions (3.4) mean that all A_1-exchange amplitudes vanish and that the natural A_2-exchange amplitudes also vanish. Only pion exchange amplitude $M^J_{0+,0-}$ contribute and they have a general form

$$M^J_{0+,0-}(s,t,m) = Q(s,t)\sqrt{2J+1}T^J(m)\sqrt{f(m)} + M^J_B(s,t,m) \tag{3.5}$$

where $T^J(m)$ are the $\pi\pi \to \pi\pi$ partial wave amplitudes with isospin decomposition

$$T^J = T^J_{I=1} \text{ for } J \text{ odd} \tag{3.6}$$

$$T^J = \frac{2}{3}T^J_{I=0} + \frac{1}{3}T^J_{I=2} \text{ for } J \text{ even}$$

in reaction $\pi^+\pi^- \to \pi^+\pi^-$. In (3.5) the function $f(m)$ is a phenomenological function that is supposed to account for absorption, and initial-state and final-state interactions. In practice one puts $f(m) = 1$. The function $Q(s,t)$ factorizes the s- and t-dependence. The term $M^J_B(s,t,m)$ is a background.

Taking into account the factor $qK(s)$ in (2.9) and the equation (3.3), Pišút and Roos arrive at a resonant parametrization of reaction cross-section

$$\frac{d^2\sigma}{dmdt} = q(2J+1)(\frac{m}{q})^2\frac{m_R^2\Gamma^2}{(m^2-m_R^2)^2+m_R^2\Gamma^2}f(m)\mathcal{N}(s,t) \tag{3.7}$$

$$+ \text{ background terms}$$

Averaging over t over an interval $< t_1, t_2 >$ gives a shape formula for the mass distribution

$$I(s,m) = q(2J+1)(\frac{m}{q})^2\frac{m_R^2\Gamma^2}{(m^2-m_R^2)^2+m_R^2\Gamma^2}f(m)N(s) \tag{3.8}$$

$$+ \text{ background terms}$$

where

$$N(s) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \mathcal{N}(s,t)dt = \frac{K(s)}{t_2 - t_1} \int_{t_1}^{t_2} |Q(s,t)|^2 dt \tag{3.9}$$

Setting $f(m) = 1$ and ignoring the background we get the Pišút-Roos resonance shape formula[39] for t-averaged mass distribution

$$I(m) = NqF(m)|BW|^2 \tag{3.10}$$

where N is the normalization constant, q is the phase space factor, $F(m)$ is the Pišút-Roos shape factor

$$F(m) = (2J+1)(\frac{m}{q})^2 = \frac{4(2J+1)}{1 - (\frac{2\mu}{m})^2} \tag{3.11}$$

and BW is the Breit-Wigner amplitude

$$BW = \frac{m_R \Gamma}{m_R^2 - m^2 - i m_R \Gamma} \tag{3.12}$$

The Pišút-Roos resonance shape formula (3.10) has been extensively used to fit partial wave intensities in previous amplitude analyses of $\pi N \to \pi^+\pi^- N$ on polarized targets (Ref. 14, 15, 16 and 23).

B. Phenomenological resonance shape formula.

In general, the experimental distribution $I(m)$ in a certain mass region is fitted to a functional form[40]

$$I(m) = \alpha_R I_R(m, m_R, \Gamma) + \alpha_B I_B(m) \tag{3.13}$$

where α_R and α_B give the fractions of resonant contribution and incoherent background. Normally I_R is taken as a square of the Breit-Wigner amplitude multiplied by a phase space factor. A coherent term may be added to the Breit-Wigner amplitude, typically a constant term. In general, the background $I_B(m)$ is a polynomial.

In the case of $\pi^- p \to \pi^-\pi^+ n$ reaction, the relevant phase space factor is just the pion momentum q in the $\pi^+\pi^-$ c.m. system and one can write for mass distributions in this reaction a phenomenological resonance shape formula

$$I(m) = Nq(m)\{|BW|^2 + B\} \tag{3.14}$$

where N is overall normalization factor and B is the background term. We can take $B = 0$ or B =constant. When $B = 0$, the phenomenological shape formula (3.14) is obtained from Pišút-Roos resonance shape formula (3.10) by setting their shape factor $F \equiv 1$. We see from (3.11) that Pišút-Roos formula (3.10) converges to phenomenological formula (3.14) for large m when background $B = 0$.

IV. The mass and width of $\sigma(750)$ from fits to S-wave amplitude $|\overline{S}|^2\Sigma$.

As seen in Fig. 1 and 2, the Monte Carlo method and the χ^2 method yield very similar results for the S-wave amplitudes $|\overline{S}|^2\Sigma$ and $|S|^2\Sigma$ in $\pi^- p \to \pi^-\pi^+ n$ at 17.2 GeV/c and for $-t = 0.005 - 0.20$ (GeV/c)^2. Both methods show that the amplitude $|\overline{S}|^2\Sigma$ resonates in both solutions while the amplitude $|S|^2\Sigma$ is non-resonating in both solutions. The Monte Carlo results appear to be smoother than the χ^2 results. The Monte Carlo method found no physical solution at mass bin 890 MeV. The solution found by χ^2 method at this mass is far off from the general trend of data in solution 1 for $|\overline{S}|^2\Sigma$. For these reasons the mass bin 880–900 MeV was excluded from the fits to $|\overline{S}|^2\Sigma$.

To determine the best values of the mass and width of $\sigma(750)$ state from the mass distribution of the resonating amplitude $|\overline{S}|^2\Sigma$ we used 4 types of fitting approaches and used a χ^2 criterion to determine the best fits. In the first approach we used a single Breit-Wigner fit. In the second approach we added an incoherent constant background to the single Breit-Wigner. In the third and fourth approaches we used two different versions of constant coherent background. In each approach we used both Pišút-Roos and phenomenological resonance shape formula and found they give very similar results. The inclusion of background leads to the narrowing of the width of $\sigma(750)$. The best χ^2 solution is obtained by the fourth approach leading to a conclusion that $\sigma(750)$ is a narrow state with a width about 100 MeV. The fitting was done using the CERN optimization program FUMILI.[45]

A. Single Breit-Wigner fit.

In this approach the mass distribution $|\overline{S}|^2\Sigma$ is fitted to a formula

$$|\overline{S}|^2\Sigma = qFN_S|BW|^2 \tag{4.1}$$

where q is the phase space factor (2.6a). The factor F is equal either

$$F = (2J+1)(\frac{m}{q})^2 \tag{4.2a}$$

for Pišút-Roos shape formula or

$$F = 1. \tag{4.2b}$$

for the phenomenological shape formula. BW is the Breit-Wigner amplitude

$$BW = \frac{m_R\Gamma}{m_R^2 - m^2 - im_R\Gamma} \tag{4.3}$$

where m_R is the resonant mass. The mass dependent width $\Gamma(m)$ depends on spin J and has a general form

$$\Gamma = \Gamma_R(\frac{q}{q_R})^{2J+1}\frac{D_J(q_R r)}{D_J(qr)} \tag{4.4}$$

In (4.4) $q_R = q(m = m_R)$ and D_J are the centrifugal barrier functions of Blatt and Weishopf:[46]

$$\begin{aligned} D_0(qr) &= 1.0 \\ D_1(qr) &= 1.0 + (qr)^2 \end{aligned} \tag{4.5}$$

where r is the interaction radius.

The results of the fit are shown in Fig. 3 and 4 for the Pišút-Roos and phenomenological shape formulas, respectively. The corresponding curves for both shape formulas are nearly identical. The numerical results are presented in Table 1. The fits to $|\overline{S}|^2\Sigma$ obtained by χ^2 method have significantly higher values of χ^2/d.o.f. However both methods give a σ mass in the range 730–750 MeV and a width in the range 230–250 MeV. Only the solution 1 of the χ^2 method gives a lower width around 190 MeV.

An important feature of the fits to $|\overline{S}|^2\Sigma$ with single Breit-Wigner formula noticeable in Fig. 3 and 4 is that all fits lie below the maximum values of the mass distributions for each solution and the method of analysis. This inability of the single Breit-Wigner formula to reproduce the resonant shape of the amplitude $|\overline{S}|^2\Sigma$ suggests that background contributions are important and their effect on the mass and width of σ state should be investigated, at least approximatively.

B. Breit-Wigner fit with incoherent background.

In this case we fit the mass distribution for $|\overline{S}|^2\Sigma$ to a formula

$$|\overline{S}|^2\Sigma = qFN_S\{|BW|^2 + B\} \tag{4.6}$$

where B is the incoherent background added to the Breit-Wigner formula (4.1). In general, B is a polynomial in m. However, since we have only 14 data points in the resonant mass range of 600–880 MeV, we will take $B =$constant.

The results of the fit are shown in Fig. 5 for the phenomenological shape formula ($F = 1$). The results with Pišút-Roos shape formula are very similar. The numerical results are given in Table 2. We notice a dramatic improvement of the fit to the solution 2 for both methods which yields a better χ^2/dof and a narrow width of about 100 MeV. There is also some improvement of the fit to the solution 1 in particular for the χ^2 method solution. This improvement is again associated with a lower χ^2/dof and a narrower width of 202 MeV and 147 MeV for the Monte Carlo and χ^2 methods, respectively. The mass of the σ state remains in the range of 730–745 MeV.

While the fits to solutions 2 are now much improved, the fits to solutions 1 are still not satisfactory with the fitted curves still below the maximum values of these mass distributions. To make further progress we turn to coherent background contributions.

C. Breit-Wigner fit with coherent background.

The nonresonant behaviour of the amplitude $|S|^2\Sigma$ (recoil nucleon transversity down) strongly suggest the presence of a coherent nonresonating background. A part of coherent background also comes from the contribution of isospin $I = 2$ amplitudes (see eq. (3.6)) which we neglected in the single Breit-Wigner fit. To understand the origins of the coherent background and to discuss its form for fits to $|\overline{S}|^2\Sigma$ it is useful to express the unnormalized moduli of S-wave transversity amplitudes in terms of unnormalized helicity amplitudes. Using (2.13) we write

$$\begin{aligned} |S|^2\Sigma &= \frac{1}{2}|S_0 + iS|^2\Sigma = qF|F_0 + iF_1|^2 \\ |\overline{S}|^2\Sigma &= \frac{1}{2}|S_0 - iS_1|^2\Sigma = qF|F_0 - iF_1|^2 \end{aligned} \tag{4.7}$$

where F_0 and F_1 are unnormalized S-wave helicity amplitudes. The terms qF have the same meaning as in (4.1) and anticipate the use of (4.7) for Breit-Wigner fits to mass distribution of $|\overline{S}|^2\Sigma$. Near the resonance with mass m_R we assume the following form of the helicity amplitudes

$$F_n(s,t,m) = R_n(s,t,m)BW(m) + B_n(s,t,m) \tag{4.8}$$

where $n = 0, 1$ is the nucleon helicity flip, BW is the Breit-Wigner amplitude (4.3), $R_n(s, t, m)$ is the pole term and $B_n(s, t, m)$ is the nonresonating background which includes the contribution from the nonresonating isospin $I = 2$ amplitudes. The energy variable s is fixed and will be omitted in the following. Since the experimental mass distributions are averaged over broad t-bins, we will eventually average also over the momentum transfer variable t. With the notation $\epsilon = \pm 1$, we can then write (4.7) in a compact form as follows

$$F_0 + i\epsilon F_1 = R_\epsilon(t, m) BW(m) + B_\epsilon(t, m) \tag{4.9}$$

where $R_\epsilon = R_0 + i\epsilon R$, and $B_\epsilon = B_0 + i\epsilon B$. It is useful to factor out the phase of R_ϵ and define

$$\begin{aligned} R_\epsilon &= |R_\epsilon| e^{i\phi_\epsilon} \\ C_\epsilon &= B_\epsilon e^{-i\phi_\epsilon} \end{aligned} \tag{4.10}$$

Then (4.9) takes the form

$$F_0 + i\epsilon F_1 = \{|R_\epsilon| BW + C_\epsilon\} e^{i\phi_\epsilon} \tag{4.11}$$

and the moduli squared of (4.7) read

$$\begin{aligned} |F_0 + i\epsilon F_1|^2 = |R_\epsilon|^2 |BW|^2 + (\mathrm{Re} C_\epsilon)^2 + (\mathrm{Im} C_\epsilon)^2 + \\ +2|R_\epsilon| \{\mathrm{Re} C_\epsilon \mathrm{Re} BW + \mathrm{Im} C_\epsilon \mathrm{Im} BW\} \end{aligned} \tag{4.12}$$

We now recall that

$$\begin{aligned} \mathrm{Re} BW &= (\frac{m_R - m^2}{m_R \Gamma}) |BW|^2 \equiv w |BW|^2 \\ \mathrm{Im} BW &= |BW|^2 \end{aligned} \tag{4.13}$$

Hence

$$\begin{aligned} |F_0 + i\epsilon F_1|^2 = \{|R_\epsilon|^2 + 2|R_\epsilon| \mathrm{Re} C_\epsilon w + 2|R_\epsilon| \mathrm{Im} C_\epsilon\} |BW|^2 \\ + (\mathrm{Re} C_\epsilon)^2 + (\mathrm{Im} C_\epsilon)^2 \end{aligned} \tag{4.14}$$

Since the amplitude $|S|^2 \Sigma(\epsilon = +1)$ does not show a clear resonant behaviour (Fig. 1 and 2), we can conclude from (4.14) that the sum of terms in the parentheses must

be small or zero. This most likely means that $|R_+|$ is small or zero implying that the pole terms in helicity amplitudes are related approximately as $R_0 \approx -iR_1$.

For the resonating amplitude $|\overline{S}|^2\Sigma(\epsilon = -1)$ the second and third terms in the parentheses in the equation (4.14) represent the effect of coherent background. In general the functions $|R_-|$ and C_- will depend on both t and m. Since these functions are not known and since we have only 14 data points in the resonance mass region 600–880 MeV, we will work in the approximation of constant background. At this point there are two possibilities.

(1) We assume that $|R_-|$ and C_- are constants independent of t and m. In this case no averaging over t is necessary and we can write (4.14) in the form

$$|\overline{S}|^2\Sigma = qFN_S\{[1 + 2wB_1 + 2B_2]|BW|^2 + B_1^2 + B_2^2\} \tag{4.15}$$

where

$$N_S = |R_-|^2, \quad B_1 = \frac{\mathrm{Re}C_-}{|R_-|}, \quad B_2 = \frac{\mathrm{Im}C_-}{|R_-|} \tag{4.16}$$

This possibility is equivalent to assuming that the constant parts of $|R_-|$ and C_- dominate in the resonant mass range 600–880 MeV. We also notice that in this case the incoherent part $B_1^2 + B_2^2$ is correlated with the coherent contribution on $(2wB_1 + 2B_2)|BW|^2$ in the formula (4.16) through the common parameters B_1 and B_2.

(2) In the second possibility, we assume that $|R_-|$ and C_- are both dependent on t and m. In this case we must average (4.14) over t over the experimentally measured interval $< t_1, t_2 >$. The averaging of (4.14) over t yields

$$|\overline{S}|^2\Sigma = qF\{[r + 2wa + 2b]|BW|^2 + c\} \tag{4.17}$$

where

$$\begin{aligned} &r =< |R_-|^2 >,\ a =< |R_-|\mathrm{Re}C_- > \\ &b =< |R_-|\mathrm{Im}C_- >,\ c =< (\mathrm{Re}C_-)^2 + (\mathrm{Im}C_-)^2 > \end{aligned} \tag{4.18}$$

In (4.18) the symbol $< >$ represents averaging over t over interval $< t_1, t_2 >$. In general, the functions r, a, b, c will depend on the mass m. Since we do not know these functions, we will assume constant values. But then there is no distinction

between r and $2b$ which can be combined into one parameter $N_S = r + 2b$ as they are two constants in a sum. Then (4.17) has the form

$$|\overline{S}|^2\Sigma = qFN_S\{[1 + 2wB_1]|BW|^2 + B\} \tag{4.19}$$

where $B_1 = b/N_S$ and $B = c/N_S$ are the coherent and incoherent contributions to the resonance shape formula. This approximation is equivalent to assumption that the functions $|R_-|$ and C_- depend mostly on t and only weakly on m. Notice that in this case the incoherent contribution B is not correlated with the coherent contribution as the parameters B and B_1 are independent.

We will refer to the first possibility (1) as Breit-Wigner fit with constant coherent background and to the second possibility (2) as the Breit-Wigner fit with t-averaged constant coherent background.

The results of the Breit-Wigner fit with constant coherent background are shown in Fig. 6 and Table 3. The results of the Breit-Wigner fit with the t-averaged constant coherent background are given in Fig. 7 and Table 4. Both Figures and Tables refer to the phenomenological shape formula with $F = 1$. The results with Pišút-Roos resonance shape formula (F given by (4.2a)) are very similar for the masses and widths although there are some differences in the fitted values of the constants B_1, B_2 or B_1 and B.

An inspection of Figures 6 and 7 show much improved fits to the data on mass distribution of $|\overline{S}|^2\Sigma$. The overall best fit (as judged by the lowest values of χ^2/dof) is provided by the Breit-Wigner fit with the t-averaged constant coherent background. However the improvements in χ^2/dof appear only in solution 1 of Monte Carlo method and solution 2 of the χ^2 method. Again, the Monte Carlo method achieves beter values of χ^2/dof compared to the χ^2 method of amplitude analysis.

The improvements in the fits brought about by the inclusion of coherent background have important consequences for the fitted values of the mass and width of $\sigma(750)$ state. From Tables 3 and 4 we find that the mass of σ in solution 1 is about 30 MeV higher than the σ mass found in solution 2. The Monte Carlo method gives the best value of σ mass 774 MeV in solution 1 and 744 MeV in solution 2 (Table 4). The χ^2 method gives the best value of σ mass 761 MeV in solution 1 and 733 MeV in solution 2 (Table 4). The data on polarized target cannot distinguish these two solutions. Since the two masses are close, we can work with a solution average.

The solution average for σ mass is 759 $\pm$ 22 MeV for Monte Carlo method and 747 $\pm$ 16 MeV for the χ^2 method. The average over the two methods gives σ mass 753 $\pm$ 19 MeV.

The most significant effect of the inclusion of coherent background is the reduction of the value of the width of σ. The Monte Carlo method gives for the best value of σ width similar values of 101 MeV and 103 MeV in solution 1 and 2, respectively (Table 4). The χ^2 method gives for the best value of σ width 134 MeV in solution 1 and 93 MeV in solution 2 (Table 4). The data on polarized target cannot distinguish these two solutions, but the high values of χ^2/dof for χ^2 method tend to favour the values for σ width from the Monte Carlo method which has low values of χ^2/dof. The solution average for the σ width is 102 $\pm$ 61 MeV for Monte Carlo method. The solution average for the σ width is 113 $\pm$ 44 MeV for the χ^2 method. Since the error on the σ width is larger for the Monte Carlo method, the two results are essentially compatible. The average over the two methods gives σ width 108 $\pm$ 53 MeV.

In conclusion, we propose to adopt the solution and method averages from the best fit values of Table 4 as the standard values of mass and width of the σ state. The obtained values are

$$m_\sigma = 753 \pm 19 \text{ MeV} \quad , \quad \Gamma_\sigma = 108 \pm 53 \text{ MeV} \tag{4.20}$$

V. The interference with $f_0(980)$ in fits to amplitude $|\overline{S}|^2\Sigma$

Törnqvist suggested[47] that the interference of $\sigma(750)$ with $f_0(980)$ resonance could influence the determination of resonance parameters of $\sigma(750)$. In the old phase shift analyses (obtained using the invalid assumption of absence of A_1-exchange), the resonance $f_0(980)$ plays an important role of smoothly interpolating the "Down" solution for δ_0^0 below 900 MeV with the results for δ_0^0 above 1000 MeV.

We will now investigate the effect of interference of $\sigma(750)$ with $f_0(980)$ on the determination of resonance parameters of $\sigma(750)$. We will find that the effect is very small. This is consistent with the fact that $f_0(980)$ is a very narrow resonance and it is positioned sufficiently far away from the narrow and strong resonance $\sigma(750)$.

The experimental data in the $f_0(980)$ mass region are given in the CERN-Munich analysis[14] of $\pi^- p \to \pi^- \pi^+ n$ on polarized target at 17.2 GeV/c for dipion masses 600–1800 MeV. From Fig. 2 and Fig. 6 of Ref. 14 it is possible to reconstruct

the amplitudes $|\overline{S}|^2\Sigma$ and $|S|^2\Sigma$. The two solutions are shown in Fig. 8. The amplitude $|\overline{S}|^2\Sigma$ resonates at 750 MeV in solution 1 and at 800 MeV in solution 2. It shows a high value at 960 MeV and a pronounced dip at 1000 MeV, indicating an interference of $f_0(980)$ with background in this mass region around 1000 MeV. The structures are less dramatic in $|S|^2\Sigma$ which does not show $\sigma(750)$ but a dip at 1000 MeV is still observable.

To proceed, we extend our parametrization (4.8) of $|\overline{S}|^2\Sigma$ to include $f_0(980)$ resonance. Recall from (4.7) that $|\overline{S}|^2\Sigma = qF|F_0 - iF_1|$. Now we write for the helicity amplitudes F_0 and F_1

$$F_n = F_n^{(\sigma)}(s,t,m)BW_\sigma(m) + R_n^{(f)}(s,t,m)BW_f(m) + B_n(s,t,m) \tag{5.1}$$

where index σ refers to $\sigma(750)$ and f refers to $f_0(980)$.
Then

$$F_0 - iF_1 = R_\sigma(s,t,m)BW_\sigma + R_f(s,t,m)BW_f + B(s,t,m) \tag{5.2}$$

Assuming that the coefficients R_σ, R_f and the background B are independent of t and m, we get an extension of the parametrization (4.15)

$$|\overline{S}|^2\Sigma = qFN_S\{[1 + 2w_\sigma B_1 + 2B_2]|BW_\sigma|^2 + \tag{5.3}$$

$$+B_1^2 + B_2^2 + [C_1^2 + C_2^2]|BW_f|^2 +$$

$$+2[(w_\sigma|BW_\sigma|^2 + B_1)(w_f C_1 - C_2) +$$

$$+(|BW_\sigma|^2 + B_2)(C_1 + w_f C_2)]|BW_f|^2\}$$

where

$$w_R = \frac{m_R^2 - m^2}{m_R \Gamma} \quad , \quad \Gamma = \Gamma_R(\frac{q}{q_R}) \quad , \quad R = \sigma, f \tag{5.4}$$

If we assume that R_σ, R_f and B depend on t and perform t-averaging, the extension of parametrization (4.19) then reads

$$|\overline{S}|^2\Sigma = qFN_S\{[1 + 2w_\sigma B_1]|BW_\sigma|^2 + B +$$

$$+2[w_\sigma B_1 + w_\sigma(w_f C_1 - C_2)|BW_f|^2 +$$

$$+(C_1 + w_f C_2)|BW_f|^2]|BW_\sigma|^2 +$$

$$+(D_1 + w_f D_2)|BW_f|^2\} \tag{5.5}$$

In the above parametrizations (5.3) and (5.5) the coefficients N_S, B_1, B_2, (B), C_1, C_2, D_1, D_2 are real constants. The data between 900 and 1120 MeV exist only in 40 MeV mass bins. Thus there is not enough data to fit the resonance parameters of $f_0(980)$. Instead we fix the mass of $f_0(980)$ at 980 MeV and its width at 48 MeV in the Breit-Wigner amplitude BW_f. Also, in our fits we took for $|\overline{S}|^2\Sigma$ below 880 MeV the results from our Monte Carlo analysis (in 20 MeV bins) and between 900 and 1120 MeV we took the results of CERN-Munich analysis (in 40 MeV bins) from Fig. 8.

The two parametrizations (5.3) and (5.5) yield virtually identical fits from 600 to 1120 MeV and the same values for mass and width of $\sigma(750)$. The fit for parametrization (5.3) is shown in Fig. 9 and the numerical values of the parameters are given in Table 5. There is a small improvement of χ^2/dof in Solution 1 which shows a better fit with the $f_0(980)$ interference. Comparison with the corresponding Table 3 shows a small increase in the mass of σ in both solutions. There is a decrease of the σ width in solution 1 from 114 MeV to 95 MeV and an increase in σ width in Solution 2 from 104 MeV to 135 MeV. The solution averages are

$$m_\sigma = 768 \pm 22 \text{ MeV} \qquad , \qquad \Gamma_\sigma = 115 \pm 38 \text{ MeV} \tag{5.6}$$

The effect of $f_0(980)$ interference is thus a small increase of average mass and width of σ as compared to values in (4.20). It is not possible to claim[47] that the low mass and the narrow width of $\sigma(750)$ are artifacts due to the neglect of interference of $\sigma(750)$ with $f_0(980)$ in our fits.

Both fits reproduce well the σ resonance peaks below 880 MeV in both solutions and the interference patterns between 920 and 1120 MeV. Particularly noteworthy in Fig. 9 is the dramatic drop in $|\overline{S}|^2\Sigma$ between 960 and 1000 MeV due to destructive interference of $f_0(980)$ with the background. The good fit in this region suggests that the assumption of constant coherent background and resonance couplings is a good approximation.

We have also attempted to fit the whole mass region of 600–1520 MeV using a three resonance parametrization with a constant background and resonance couplings. The fit was not successful as the $f_0(1300)$ resonance was not well reproduced. This indicates that the background above 1120 MeV is different and the assumption of constant background for such a large mass range does not work. Next we fitted the $f_0(1300)$ resonance in the mass range of 1120 to 1520 MeV to a single

Breit-Wigner with incoherent background. The results are shown in Fig. 9 for the two solutions (differing in values of $|\overline{S}|^2\Sigma$ at 1480 MeV). The Solution 1 above 1120 MeV connects smoothly with both solutions below 1120 MeV while the Solution 2 shows a small discontinuity at 1120 MeV. Surprisingly, the incoherent background in both solutions is consistent with zero. This again indicates that above 1100–1200 MeV the background (if any) is different from the low mass region below 1100 MeV. The numerical results of the fit to $f_0(1300)$ in the mass region 1120–1520 MeV are given in Table 6. We note the similarity of mass and width of resonances $f_0(1300)$ and $f_2(1270)$.

VI. The mass and width of $\sigma(750)$ state from the fits to S-wave intensity I_S.

Previous amplitude analyses[13–15,23] of $\pi^- p \to \pi^-\pi^+ n$ and $\pi^+ n \to \pi^+\pi^- p$ data on polarized targets fitted only certain partial wave intensities using Pišút-Roos resonance shape formula without any background. It is of interest to perform Breit-Wigner fits to the S-wave intensity I_S and compare the results with the results of fits to resonating amplitude $|\overline{S}|^2\Sigma$ in $\pi^- p \to \pi^-\pi^+ n$ at 17.2 GeV/c. Because of lower statistics, data for $\pi^+ n \to \pi^+\pi^- p$ at 5.98 and 11.85 GeV/c allow fits only to S-wave intensity I_S. This is thus our primary aim in fitting S-wave intensity: to extract information about the mass and width of σ in $\pi^+ n \to \pi^+\pi^- p$ reaction measured at larger momentum transfers $-t = 0.2 - 0.4$ (GeV/c)^2.

Let us recall that the S-wave intensity I_S is defined as

$$I_S(s,t,m) = (|S|^2 + |\overline{S}|^2)\Sigma = (|S_0|^2 + |S_1|^2)\Sigma \qquad (6.1)$$

Since there are two independent solutions for the amplitudes $|S|^2$ and $|\overline{S}|^2$, there are 4 solutions for the S-wave intensity. We label these 4 solutions as $I_S(1,1), I_S(1,2), I_S(2,1)$ and $I_S(2,2)$ where

$$I_S(i,j) = (|S(i)|^2 + |\overline{S}(j)|^2)\Sigma \quad , \quad i,j = 1,2 \qquad (6.2)$$

The results for the 4 solutions of I_S obtained by the Monte Carlo amplitude analysis are shown in Fig. 10. The results for $I_S(1,1)$ and $I_S(2,2)$ obtained by the χ^2 minimization method are shown in Fig. 11. Again, there is a remarkable agreement between the results of these two different methods of analysis. The solutions $I_S(1,1)$

and $I_S(2,1)$ are clearly resonating but the solutions $I_S(1,2)$ and $I_S(2,2)$ do not show a clear resonant behaviour. This is due to the large nonresonating contribution from the amplitude $|S|^2\Sigma$ (see Fig. 1 and 2). The amplitude $|S|^2\Sigma$ represents a nontrivial nonresonating background in all four solutions and is thus expected to distort the results of Breit-Wigner fits to I_S.

We first performed fits to I_S using a single Breit-Wigner formula without any background

$$I_S = qFN_S|BW|^2 \tag{6.3}$$

In all fits to S-wave intensities we used the Pišút-Roos shape factor $F = (2J+1)(m/q)^2$. The results are shown as solid lines in Fig. 10 and 11 and in Tables 7 and 8 for Monte Carlo and χ^2 methods, respectively. We notice in Fig. 10 and 11 that the single Breit-Wigner fit is well below the maximum values of the mass distribution I_S. In Monte Carlo analysis the mass of σ is around 766 MeV in all four solutions. The width is around 260 MeV for the first 3 solutions and is larger at 303 MeV for the solutions $I_S(2,2)$. In χ^2 method the σ mass and width in solution $I_S(1,1)$ is in agreement with Monte Carlo results, but the width of $I_S(2,2)$ is larger at 408 MeV and also mass is higher at 786 MeV.

Next we performed Breit-Wigner fit with a constant incoherent background using a formula

$$I_S = qFN_S\{|BW|^2 + B\} \tag{6.4}$$

where B is the constant background term. The results are shown as dashed lines in Fig. 10 and 11 and in Tables 7 and 8 for the Monte Carlo and χ^2 methods, respectively. While the masses of σ remain the same, there is a general reduction of the width of σ associated with improved fits to the data and lower values of χ^2/dof. In Monte Carlo method the width of σ is reduced to 210 MeV in the first 3 solutions to I_S. However the most dramatic and unexpected change occurs in the solution $I_S(2,2)$ in both methods. There is a considerable improvement in the fit to the data and the width is drastically reduced to 188 MeV in both methods indicating the existence of a narrow σ state even in the broad looking mass distribution.

The best determination of σ width from the fits to S-wave intensity I_S is still double of the best value obtained in fits directly to the amplitude $|\overline{S}|^2\Sigma$ (Table 4). This discrepancy shows that the determination of resonance parameters from the spin-averaged intensities is not fully reliable when there is a presence of large

nonresonating nontrivial background as is the case of the amplitude $|S|^2\Sigma$. The characteristic feature of this situation is that the S-wave intensity does not show a clear reonant structure in all four solutions.

This situation does not occur in the data on S-wave intensity in $\pi^+ n \to \pi^+\pi^- p$ at larger momentum transfers $-t = 0.2 - 0.4$ (GeV/c)^2. The results from Monte Carlo amplitude analysis are shown in Fig. 12 and 13 at 5.98 and 11.85 GeV/c, respectively. We notice that all four solutions at both energies show clear resonant structures. This suggests that the determination of resonance parameters from S-wave intensities at these momentum transfers should be more reliable. However, this advantage is somewhat offset by the lower statistics of the data and large errors.

We have again performed fits using single Breit-Wigner formula (6.3) and the Breit-Wigner fit with constant incoherent background using formula (6.4). The results are shown in Fig. 12 and 13 and in Tables 9 and 10 for incident momenta of 5.98 and 11.85 GeV/c, respectively. The fit with constant background (dashed lines) is a clear improvement over a single Breit-Wigner fit (solid lines). The improvement of the fit with the constant background is again associated with lower values of χ^2/dof and with reduction of the width of σ in all solutions at both energies. However, there are differences in values for the mass and the width of σ between the solutions as well as between energies. At 5.98 GeV/c, the mass ranges from 706 to 745 MeV and the width ranges from 145 to 262 MeV. At 11.85 GeV/c, the mass is higher and ranges from 756 to 782 MeV while the width is lower ranging from 117 to 202 MeV. The differences are probably due to lower statistics.

The solution averages for the mass and width of σ from fits to I_S are as follows: At 5.98 GeV/c

$$m_\sigma = 730 \text{ MeV} \pm 27 \text{ MeV} \quad , \quad \Gamma_\sigma = 195 \pm 81 \text{ MeV} \tag{6.5}$$

At 11.85 GeV/c

$$m_\sigma = 768 \pm 17 \text{ MeV} \quad , \quad \Gamma_\sigma = 166 \pm 54 \text{ MeV} \tag{6.6}$$

At 17.2 GeV/c

$$m_\sigma = 767 \pm 9 \text{ MeV} \quad , \quad \Gamma_\sigma = 204 \pm 75 \text{ MeV} \tag{6.7}$$

The best values for the mass and width of σ obtained from fits to the S-wave intensities at the three energies are in general agreement. The small differences are

likely due to the fact that the approximation of constant incoherent background may work differently at various energies and momentum transfers. The differences in mass of σ from the fits to $|\overline{S}|^2\Sigma$ and to I_S are small. The difference in the value of the width from the best fits to $|\overline{S}|^2\Sigma$ with coherent background and the fits to I_S are somewhat large but the results are still consistent. At 17.2 GeV/c they are due to large nonresonating contributions from the amplitude $|S|^2\Sigma$. The differences also reflect the need for inclusion of coherent background and its better description than a constant. This in turn would require more data of high statistics in the resonance region 600–900 MeV.

VII. Remarks on determinations of $\pi\pi$ phase shifts.

The amplitude analyses of measurements of $\pi N_\uparrow \to \pi^+\pi^- N$ on polarized targets provide a model-independent and solution-independent evidence for a narrow scalar state $I = 0\ 0^{++}(750)$. The question arises how to understand the absence of such a state in the conventional S-wave phase shift δ_0^0 in $\pi\pi$ scattering.[11,24–31]

Of course, there are no actual measurements of pion-pion scattering and there is no partial-wave analysis of $\pi\pi \to \pi\pi$ reactions in the usual sense. The $\pi\pi$ phase shifts are determined indirectly from measurements of $\pi^- p \to \pi^-\pi^+ n$ on unpolarized targets using several strong enabling assumptions. One of these crucial assumptions – the absence of A_1 exchange amplitudes – leads to predictions for polarized spin density matrix (SDM) elements and for the measured amplitudes, and it is thus directly testable in the measurements on polarized targets. As we shall see below, the assumption of absence of A_1-exchange amplitudes is totally invalidated by the data on polarized targets. The polarization measurements also cast some doubt on the fundamental assumption of factorization of mass m and momentum transfer t in the crucial pion exchange amplitudes. We must use the results of measurements on polarized targets to judge the validity of $\pi\pi$ phase shifts, and not vice versa. We are thus led to the conclusion that the indirect and model-dependent determinations of $\pi\pi$ phase shifts cannot be correct. This explains the absence of $I = 0\ 0^{++}(750)$ resonance in the δ_0^0 phase shift from these analyses.

We will now review the basic assumptions common to all determinations of $\pi\pi$ phase shifts.[11,24–31]

A priori, there is no connection between the partial wave amplitudes in $\pi\pi \to \pi\pi$ scattering and the production amplitudes in $\pi N \to \pi^+\pi^- N$ reactions.

We recall that in $\pi N \to \pi^+\pi^- N$ there are two S-wave production amplitudes $S(s,m,t)$ and $\overline{S}(s,m,t)$ (or $S_0(s,m,t)$ and $S_1(s,m,t)$) while in $\pi\pi \to \pi\pi$ there is one S-wave amplitude (or phase shift δ_0^0) dependent only on the energy E. Also, in $\pi N \to \pi^+\pi^- N$ there are six P-wave production amplitudes $L, \overline{L}, U, \overline{U}, N, \overline{N}$ (or $L_n, U_n, N_n, n = 0, 1$) which depend on variables s, m, t while in $\pi\pi \to \pi\pi$ there is again one P-wave amplitude (or phase shift δ_1^1) dependent only on the energy E. To make the connection between the production amplitudes in $\pi N \to \pi^+\pi^- N$ and the partial-wave amplitudes in $\pi\pi \to \pi\pi$ the following assumptions of factorization and identification are postulated in all determinations of $\pi\pi$ phase shifts from unpolarized data on $\pi N \to \pi^+\pi^- N$.

The starting point are the dimeson helicity $\lambda = 0$ pion exchange amplitudes S_1 and L_1 in the t-channel. It is assumed that the t and m dependence in these amplitudes factorizes:

$$\begin{aligned} S_1(s,m,t) &= N\frac{\sqrt{-t}}{t-\mu^2}F_0(t)\frac{m}{\sqrt{q}}f_0(m) \\ L_1(s,m,t) &= N\frac{\sqrt{-t}}{t-\mu^2}F_1(t)\frac{m}{\sqrt{q}}f_1(m) \end{aligned} \tag{7.1}$$

where t is the momentum transfer at the nucleon vertex, m and q are the dipion mass and the π^- momentum in the $\pi^-\pi^+$ c.m. frame. The form factors $f_J(t)$ describe the t-dependence and the functions $f_J(m)$, $J = 0, 1$, describe the mass dependence. N is a normalization constant. Furthermore, the functions $f_J(m)$ are assumed to be the partial-wave amplitudes in $\pi^-\pi^+ \to \pi^-\pi^+$ reaction at c.m. energy m:

$$\begin{aligned} f_0 &= \frac{2}{3}f_0^{I=0} + \frac{1}{3}f_0^{J=2} \\ f_1 &= f_1^{I=1} \end{aligned} \tag{7.2}$$

The partial wave amplitudes f_J^I with definite isospin I are defined so that in the $\pi\pi$ elastic region

$$f_J^I = \sin\delta_J^I e^{i\delta_J^I} \tag{7.3}$$

The phase shifts δ_J^I are determined from the amplitudes S_1 and L_1 which are calculated from the data on $\pi^- p \to \pi^-\pi^+ n$ on unpolarized target. However the calculation of amplitudes S_1 and L_1 from the $\pi^- p \to \pi^-\pi^+ n$ data cannot be done without additional assumptions. There is simply more amplitudes than data. To proceed

further all determinations of $\pi\pi$ phase shifts must assume that all A_1-exchange amplitudes vanish:

$$S_0 = L_0 = U_0 \equiv 0 \tag{7.4}$$

With the assumptions (7.4), two solutions for the S-wave phase shift δ_0^0 are found:[27,28] "Down" solution which is non-resonating and "Up" solution which resonates at the mass around 770 MeV with a width about 150 MeV. The resonating solution was rejected because it disagreed with the $\pi^0\pi^0$ mass spectrum from a low-statistics experiment[48] on $\pi^- p \to \pi^0\pi^0 n$ at 8 GeV/c.

There is no theoretical proof of factorization (7.1) and identification (7.2) of functions f_J with $\pi\pi$ partial-wave amplitudes. It is not obvious that the $\pi\pi$ phase shifts calculated from $\pi^- p \to \pi^-\pi^+ n$ data using the assumptions (7.1)–(7.3) would coincide with $\pi\pi$ phase shifts determined directly from real pion-pion scattering. Only such comparison could test the assumption (7.2).

The factorization (7.1) implies that the mass spectrum of amplitudes $|S_1|^2$ and $|L_1|^2$ is independent of t. This consequence of factorization can be tested in measurements of $\pi N \to \pi^+\pi^- N$ on polarized targets. In Fig. 14 we show t-evolution of mass dependence of lower and upper bounds[19] on normalized moduli $|L|^2$, $|\overline{L}|^2$, $|U|^2$ and $|\overline{U}|^2$. The data at $t = -0.068$ (GeV/c)^2 are from $\pi^- p \to \pi^-\pi^+ n$ at 17.2 GeV/c, the rest is from $\pi^+ n \to \pi^+\pi^- p$ at 5.98 GeV/c. The Fig. 14 shows a clear and pronounced dependence of mass spectra of amplitudes $|L|^2$ and $|\overline{L}|^2$ on momentum transfer t. In particular, there is a clear change of mass spectrum below $-t = 0.25$ (GeV/c)^2, a region of t relevant to determinations of $\pi\pi$ phase shifts. While this change could be entirely due to A_1 exchange amplitude L_0, this cannot be guaranteed. The factorization assumption (7.1) thus cannot be taken for granted and further tests of this assumption are required in future high statistics measurements of $\pi N \to \pi^+\pi^- N$ on polarized targets.

The assumption (7.4) of absence of A_1-exchange amplitudes has several consequences that can be directly tested in measurements on polarized targets. ¿From (2.13) we see that absence of A_1 exchange amplitudes implies

$$|A| = |\overline{A}| \qquad \text{for} \qquad A = S, L, U \tag{7.5}$$

The equality of moduli of amplitudes with the recoil nucleon transversity "down" and "up" is not observed experimentally. We can see in Fig. 1 and 2 that the S-wave

amplitudes $|S|$ and $|\overline{S}|$ are clearly unequal at 17.2 GeV/c and $-t = 0.005 - 0.20$ $(\mathrm{GeV/c})^2$. In Fig. 14 we see that the P-wave amplitudes $|L|$ and $|\overline{L}|$ are different in every t-bin from 0.005 to 0.60 $(\mathrm{GeV/c})^2$, and that the difference is largest at small t, the region of most importance to determination of $\pi\pi$ phase shifts.

The A_1-exchange is large and nontrivial also above 900 MeV and in higher partial waves D and F. This finding of CERN-Munich analysis[14] of $\pi^- p \to \pi^-\pi^+ n$ data on polarized target in the mass range 600–1800 MeV is shown in Fig. 15. The figure shows the ratios of moduli of amplitudes with recoil nucleon transversity "down" and "up" for S-, P-, D- and F-wave amplitudes with dimeson helicity $\lambda = 0$ which are directly relevant for determination of the corresponding phase shifts. The deviations from 1 indicate the strength of A_1-exchange. We can see in Fig. 15 that A_1-exchange is important in all waves up to 1800 MeV at small $-t = 0.005 - 0.20$ $(\mathrm{GeV/c})^2$. The determinations of $\pi\pi$ phase shifts above 900 MeV also assumed the absence of A_1-exchange amplitudes. We must conclude that the determinations of $\pi\pi$ phase shifts from S-wave to F-wave in the mass region from 600 to 1800 MeV are not reliable. Theoretical calculations and analyses based on these phase shifts are therefore not reliable as well.

Below 1000 MeV, where the S- and P-wave dominate, the assumptions (7.4) lead to predictions for polarized SDM elements that can be directly compared with the data. The predictions of (7.4) are[23]

$$\rho^y_{ss} + \rho^y_{00} + 2\rho^y_{11} = -2(\rho^y_{00} - \rho^y_{11}) = +2\rho^y_{1-1} \tag{7.6}$$

$$\mathrm{Re}\rho^y_{10} = \mathrm{Re}\rho^y_{1s} = \mathrm{Re}\rho^y_{0s} \equiv 0 \tag{7.7}$$

The data for polarized SDM elements clearly rule out these predictions as is shown in Figs. 16 and 17 for $\pi^- p \to \pi^-\pi^+ n$ at 17.2 GeV/c. We find that $\rho^y_{ss} + \rho^y_{00} + 2\rho^y_{11}$ and $-2(\rho^y_{00} - \rho^y_{11})$ have large magnitudes but opposite signs while $2\rho^y_{1-1}$ has a small magnitude. The interference terms $\mathrm{Re}\rho^y_{10}$, $\mathrm{Re}\rho^y_{1s}$ and $\mathrm{Re}\rho^y_{0s}$ are all dissimilar and have large nonzero values. On the basis of this evidence we again must conclude that the past determinations of $\pi\pi$ phase shifts from unpolarized data on $\pi N \to \pi^+\pi^- N$ are questionable.

The assumption of absence of A_1 exchange amplitudes means that pion production in $\pi N \to \pi^+\pi^- N$ reactions does not depend on nucleon spin. What the measurements of $\pi N \to \pi^+\pi^- N$ on polarized targets found is that the pion production depends strongly on nucleon spin. The dynamics of the pion production is not

as simple as has been assumed in the past determinations of $\pi\pi$ phase shifts. New determinations of $\pi\pi$ phase shifts are now required that do take into account the existence of A_1 exchange. Since the contributions of A_1 exchange amplitudes are large and nontrivial, the revisions of $\pi\pi$ phase shifts will be significant. The new revised S-wave phase shift δ_0^0 is then expected to show evidence for narrow scalar state $\sigma(750)$ in agreement with the measurements on polarized targets.

VIII. Questions concerning evidence for narrow $\sigma(750)$

A. Up-Down ambiguity and analyticity constraints.

Recently it has been claimed[49,50] that $\pi\pi$ phase shift δ_0^0 can be determined from the S-wave intensities I_S obtained in our amplitude analysis of $\pi^- p \to \pi^- \pi^+ n$ on polarized target at 17.2 GeV/c, and that it would show the old Up-Down ambiguity of δ_0^0. Only Up solution indicates a narrow σ state and it is excluded because it is inconsistent with Roy equations.[51] From this it was concluded that $\sigma(750)$ does not exist[49] or that the evidence must be treated with reservation.[50]

To answer this objection we first recall from (2.15) that

$$I_S = (|S_0|^2 + |S_1|^2)\Sigma \tag{7.1}$$

Here the amplitude S_1 is connected to δ_0^0 through (7.1) and (7.3), and S_0 is the unknown A_1 exchange amplitude. It is obvious from this expression that the determination of δ_0^0 from data on I_S depends on the model used for A_1 exchange amplitude S_0. The data on polarized target require large A_1 exchange amplitudes. At present the A_1 exchange amplitudes are not known. We must therefore conclude that the phase shift δ_0^0 cannot be determined from the data on S-wave intensity I_S at present.

Nevertheless, the data on I_S do tell us something very important about the solutions for δ_0^0. There are four solutions for I_S: $I_S(1,1), \ldots, I_S(2,2)$. Consequently there will be a fourfold ambiguity in δ_0^0 for any given model of A_1 exchange amplitude S_0. However, as can be seen in Fig. 10, the four solutions for I_S are all very similar quantitatively. Consequently the four solutions for δ_0^0 are expected to be very close to each other and similar. This contrasts with the large differences between the old Up and Down solutions. Fig. 18 shows S-wave intensity normalized to 1 at maximum for Down (curve A) and Up (curve B) solutions from the

typical analysis of Estabrooks et al.[27,28] The large differences between the Up and Down solutions contrast sharply with the small differences shown between S-wave intensities $I_S(1,1)$ and $I_S(2,2)$ in Fig. 18. On the basis of the similar behaviour of all solutions for I_S we do not anticipate the emergence of the old Up-Down ambiguity problem in δ_0^0. It is even possible that the small differences between the four solutions for I_S can be explained entirely as a small ambiguity in A_1 exchange amplitude S_0 leading to a unique determination of δ_0^0 from the data on polarized target.

The above discussion applies also to the determination of P-wave phase shift δ_1^1 from $I_L = (|L_0|^2 + |L_1|^2)\Sigma$. The amplitude L_1 is connected to δ_1^1 by (7.1) and (7.3) while L_0 is another unknown A_1-exchange amplitude. The four solutions for I_L are again very close so we expect similar solutions for δ_1^1.

Assuming a model for A_1 exchange amplitudes S_0 and L_0, the obtained phase shifts δ_0^0 and δ_1^1 can be tested for consistency with dispersion relations[51] (Roy equations). If an inconsistency is found it means that we have to modify our model for A_1 exchange amplitudes S_0 and L_0, and try again. It is important to realize that Roy equations do not test the validity of the experimentally measured amplitudes $|S|^2$, $|\overline{S}|^2$, $|L|^2$, $|\overline{L}|^2$ or intensities I_S and I_L. The Roy equations are constraints only on $\pi\pi$ phase shifts which follow from the analyticity properties of partial wave amplitudes in $\pi\pi \to \pi\pi$ scattering. However the requirement of consistency of phase shifts with the Roy equations can be used to constrain the possible models of A_1 exchange amplitudes.

We conclude that the experimental evidence for the narrow state $\sigma(750)$ is not in contradiction with analyticity and dispersion relations for $\pi\pi$ partial waves. The existence of A_1-exchange and narrow $\sigma(750)$ are experimental findings from measurements on polarized targets independent of the Roy equations. These experimental facts cannot be refuted by comparisons with standard phase shifts because these were obtained using an invalid assumption of absence of A_1-exchange.

B. Absence of $\sigma(750)$ in $\gamma\gamma \to \pi^+\pi^-$ and central production $pp \to pp\pi^+\pi^-$.

Morgan and Pennington suggested to discount the evidence for existence of narrow $\sigma(750)$ in $\pi N \to \pi^+\pi^- N$ because this state has not been observed in $\gamma\gamma \to \pi^+\pi^-$ reaction[49] and in central production[49,52] $pp \to pp\pi^+\pi^-$. However there are good reasons why one would not expect to observe narrow $\sigma(750)$ in these processes.

In the next section we shall argue that the narrow $\sigma(750)$ is the lowest mass scalar gluonium $0^{++}(gg)$. The principal support for this proposal is precisely the fact that $\sigma(750)$ state is not observed in $\gamma\gamma \to \pi^+\pi^-$ reaction. Since gluons do not couple directly to the photons, we expect $\sigma(750)$ not to appear in reaction $\gamma\gamma \to \pi^+\pi^-$ if it is pure gluonium or if it has only a small $q\overline{q}$ component.

The reaction $pp \to pp\pi^+\pi^-$ was measured[53] at the CERN Intersecting Storage Rings (ISR) in a search for scalar gluonium. The structures reported in the moments $H(11)$ and $H(31)$ near $m(\pi^+\pi^-) \approx 750$ MeV are consistent with $\sigma(750)$ and $\rho^0(770)$ interference.

Assuming parity conservation there are 5 S-wave amplitudes and 15 P-wave amplitudes in this reaction. The $\sigma(750)$ state may contribute only to some S-wave amplitudes and not to the others, as it does in $\pi^-p \to \pi^-\pi^+n$ with amplitudes $|\overline{S}|^2\Sigma$ and $|S|^2\Sigma$. As we see in Fig. 10, the S-wave intensity $I_S(2,2)$ does not immediately suggest existence of narrow $\sigma(750)$. With 5 S-wave amplitudes in $pp \to pp\pi^+\pi^-$ it is very likely that $\sigma(750)$ stays hidden. We can observe $\sigma(750)$ in $\pi^-p \to \pi^-\pi^+n$ and $\pi^+n \to \pi^+\pi^-p$ reactions only when these production processes are measured on polarized targets, and the S- and P-wave amplitudes can be separated in a model independent way. For the same reasons we may see $\sigma(750)$ in central production $pp \to pp\pi^+\pi^-$ only when measurements with polarized initial protons are made and the resonating S-wave amplitudes can be isolated. The ISR experiment does not separate the S- and P-wave amplitudes, and thus it is not conclusive.

C. Comparison with other results for σ state.

DM2 Collaboration measured[54] $\pi^+\pi^-$ mass distribution in $J/\psi \to \omega\pi^+\pi^-$ decays and observed a quite broad low mass resonance (see Fig. 13a of Ref. 54). Interpreted as an $I = 0$ 0^{++} σ state, a single Breit-Wigner fit gives $m_\sigma = (414 \pm 20)$ MeV, $\Gamma_\sigma = (494 \pm 58)$ MeV. There is no indication for such state in our data on S-wave intensity I_S in $\pi^+n \to \pi^+\pi^-p$ at 5.98 and 11.85 GeV/c (see Fig. 12 and 13 above). The reasons for the discrepancy are not clear at the present.

Several recent theoretical analyses[55,56,57] claimed existence of a σ meson with a mass around 1000 MeV and a broad width of 460–880 MeV. These analyses use as an input the S-wave phase shift δ_0^0 and thus neglect the A_1 exchange and other spin effects observed in pion production (see e.g. eq. (5) in Ref. 56). It is possible that when these analyses include in their fits A_1 exchange that they will find a narrow

σ in agreement with the CERN data on polarized targets.

IX. Constituent structure of the $\sigma(750)$ resonance.

In the usual quark model meson resonances are $q\overline{q}$ states. The mass of $\sigma(750)$ is too low for it to be a $q\overline{q}$ state. The mass M of the $q\overline{q}$ state increases with its angular momentum L as $M = M_0(2n+L)$ where n is the degree of radial excitation. The lowest mass scalar mesons are $^3P_0$ states with masses expected to be around 1000 MeV or higher.

It was suggested that $0^{++}(700)$ could be a four-quark $q\overline{q}q\overline{q}$ state in the MIT bag model.[58] However, more detailed studies of $q\overline{q}q\overline{q}$ systems conclude that pure multiquark hadrons do not exist[59,60] with $\pi^+\pi^-$ decay.[61] We can also exclude the possibility that $\sigma(750)$ is a hybrid state $q\overline{q}g$. The lowest mass hybrid state must be a 0^{-+} or 1^{-+} state. Calculations based on bag models, QCD sum rules, lattice QCD and a string model all estimate[62] the masses of $0^{++}(q\overline{q}g)$ states to be above 1500 MeV.

Ellis and Lanik discussed the couplings of scalar gluonium σ on the basis of the low energy theorems of broken chiral symmetry and scale invariance, implemented using a phenomenological lagrangian.[63] They obtained for $\sigma \to \pi^+\pi^-$ decay the following partial width

$$\Gamma(\sigma \to \pi^+\pi^-) = \frac{(m_\sigma)^5}{48\pi G_0} \tag{9.1}$$

where $G_0 \equiv < 0|(\alpha_s/\pi)F_{\mu\nu}F^{\mu\nu}|0 >$ is the gluon-condensate term[64] parametrizing the non-perturbative effects in QCD. The numerical values were estimated by the ITEP group[64] to be $G_0 \approx 0.012$ (GeV)^4 or up to $G_0 \approx 0.030$ (GeV)^4 in later calculations.[65,66] It is very interesting to note, that when we take $G_0 = 0.015$ (GeV)^4 the Ellis-Lanik theorem (9.1) predicts partial width of $\sigma \to \pi^+\pi^-$ decay $\Gamma = 107$ MeV for the mass $m_\sigma = 753$ MeV. This result is in perfect agreement with (4.20), the solution and method average values of mass and width of $\sigma(750)$ from the best fit to the measured mass distribution $|\overline{S}|^2\Sigma$ (Table 4). When we use for m_σ the value 768 MeV obtained in interference fits with $f_0(980)$ then Ellis-Lanik theorem predicts a width $\Gamma(\sigma \to \pi^+\pi^-) = 118$ MeV, again in perfect agreement with (5.6) where $\Gamma_\sigma = 115 \pm 38$ MeV. From this agreement we can conclude that the $\sigma(750)$ is best understood as the lowest mass gluonium state $0^{++}(gg)$.

The gluonium interpretation of $\sigma(750)$ gathers further support from the lack of observation of $\sigma(750)$ in the reactions $\gamma\gamma \to \pi^+\pi^-$ and $\gamma\gamma \to \pi^0\pi^0$. Since gluons

do not couple directly to photons we expect $\sigma(750)$ not to appear in reactions $\gamma\gamma \to \pi\pi$ if it is a pure gluonium state or if it contains only a small $q\overline{q}$ component. This conclusion is supported by the PLUTO and DELCO data.[67,68] However, the more recent DM1/2 data[69,70] show an excess over the Born term expectation that is attributed to the formation of a broad scalar resonance with a two-photon width of (10 ± 6) MeV. This would suggest some $q\overline{q}$ component in the $\sigma(750)$ state. The most recent results[71] are on $\gamma\gamma \to \pi^0\pi^0$ which show no evidence for a scalar state near 750 MeV.

Lattice QCD calculations by several groups[72−75] initially concluded that the gluonium ground state $0^{++}(gg)$ has a mass near the ρ^0 meson: 740± 40 MeV. The most recent lattice QCD calculations predict much higher mass of the lowest scalar gluonium: the UKQCD group[76] predicts 1550 ± 50 MeV while the IBM group[77,78] predicts 1740 ± 70 MeV. However, it is important to remember that these calculations are for quenched QCD so that there is no coupling of the primitive gluonium to quarks. The coupling of gluonium to two pseudoscalars may have a significant effect on the gluonium mass and width.[52]

We conclude that while the gluonium interpretation of the $\sigma(750)$ state is in agreement with low energy theorems of broken chiral symmetry and scale invariance, it is at variance with the most recent lattice QCD calculations. It is necessary to study this discrepancy and understand its origins and implications.

Finally we note that the anomalous energy dependence of pp and np elastic polarizations and the departure from the mirror symmetry in πN elastic polarizations at intermediate energies require a low-lying Regge trajectory[79,80] corresponding to $\sigma(750)$. These anomalous structures in the polarization data may have been the first evidence for a gluonium exchange in two-body reactions.

X. Amplitude spectroscopy – a new direction in hadron spectroscopy.

The vast majority of hadron resonances have been identified through study of mass distributions of spin averaged cross-sections. Experiments with polarized targets opened a whole new approach to experimental hadron spectroscopy by making accessible the study of hadron production on the level of spin dependent production amplitudes. We may refer to this new approach to detecting and studying hadron resonances as amplitude spectroscopy.

This work represents the first effort to determine resonance parameters directly

from a measured spin dependent production amplitude. In this case it was the amplitude $|\overline{S}|^2\Sigma$ measured in $\pi^- p_\uparrow \to \pi^-\pi^+ n$ at 17.2 GeV/c. We have found that the best Breit-Wigner fits to the resonance $\sigma(750)$ on the amplitude level differ markedly from the Breit-Wigner fits to the spin-averaged S-wave intensities, and provide a more reliable information about the resonance parameters.

However, the significance of amplitude spectroscopy goes far beyond the more precise determinations of resonance parameters. The amplitude spectroscopy opens several prospects for new physics:

(a) Let us call dominant resonances those states which can be experimentally identified in spin-averaged cross-sections or Dalitz plots. Historically these dominant resonances led to the standard quark model and to QCD. However, a new species of subdominant resonances may exist which can be identified only at the level of spin dependent production amplitudes. The existence and properties of subdominant resonances could reveal new components of hadron structure to which unpolarized experiments are totally blind and could lead us beyond the standard quark model and standard QCD.

(b) The production of resonances (dominant and subdominant) may depend on nucleon spin and this dependence will provide important information about the dynamics of hadron interactions and about the very nature of hadron resonances.

(c) Standard QCD predicts new kinds of resonances such as dibaryon, gluonium and hybrid states. Many of these states may not be observable in the spin-averaged measurements which could explain the limitted success in identifying these states so far.

The $\sigma(750)$ is the first example of a subdominant resonance observable only on the level of measured spin dependent production amplitudes. We have interpreted this resonance as the lowest mass gluonium $0^{++}(gg)$ in the Section IX. However, we must be open also to the possibility that $\sigma(750)$ represents the first signal of a new physics beyond the standard quark model and QCD. It may indicate the existence of a new component in hadron structure.

The measurements[18] of $K^+n \to K^+\pi^- p$ reactions at 5.98 and 11.85 GeV/c at CERN-PS also allowed a model independent amplitude analysis[81] of this reaction at 5.98 GeV/c. The results of the amplitude analysis are in excellent agreement with Additive Quark Model predictions.[82] The data also suggest[18,81] the existence

of another subdominant resonance, $I = \frac{1}{2}\ 0^{++}(890)$ with a narrow width of about 20 MeV. Such new resonance $\kappa(890)$ under $K^{0*}(892)$ could also signal new physics beyond the standard QCD. One such possibility is discussed in Ref. 81.

We recall that complete measurements of spin observables in two-body and quasi two-body reactions enable construction of spin amplitudes. The spin amplitudes show dip structures in their moduli associated with rapid and large changes of their relative phases. These dip structures resemble absorption resonances and systematic study of dip structures in two-body and other exclusive processes like $\pi N \to \pi^+\pi^- N$ is a natural extension of hadron spectroscopy into the space-like region. Study of time-like resonances and space-like dips in spin dependent amplitudes should bring entirely new insights into the hadron dynamics and structure.

To explore these new frontiers of hadron spectroscopy and hadron dynamics, new advanced hadron facilities dedicated in large part to measurements with spin will be required. The proposed Canadian KAON Factory,[83] Los Alamos Hadron Facility[84] and European Hadron Facility[85] could integrate the new advanced technologies of polarized beams and targets in a single spin physics facility[86] to systematically advance the exploration and development of these new frontiers in hadron physics.[86–88]

XI. Summary

The measurements of reactions $\pi^- p_\uparrow \to \pi^-\pi^+ n$ at 17.2 GeV/c and $\pi^+ n_\uparrow \to \pi^+\pi^- p$ at 5.98 and 11.85 GeV/c on polarized target provide model-independent and solution-independent evidence for a narrow scalar state $\sigma(750)$. The amplitude analyses of $\pi^- p_\uparrow \to \pi^-\pi^+ n$ at small t using χ^2 minimization method[13] and Monte Carlo method[23] yield very similar results for moduli of transversity amplitudes and cosines of their relative phases. In particular they agree that the transversity "up" S-wave amplitude $|\overline{S}|^2\Sigma$ resonantes near 750 MeV while the transversity "down" amplitude $|S|^2\Sigma$ is nonresonating and constitutes a large background in the spin-averaged S-wave intensity $I_S = (|S|^2 + |\overline{S}|^2)\Sigma$. For this reason it is preferable to determine resonance parameters of $\sigma(750)$ directly from the measured mass distribution of $|\overline{S}|^2\Sigma$.

We have performed several types of Breit-Wigner fits to $|\overline{S}|^2\Sigma$. We have shown that the Pišút-Roos resonance shape formula and phenomenological shape formula give similar results. Single Breit-Wigner fits yield a width of $\sigma(750)$ in the range

192–256 MeV. We have studied the effect of background in three approaches: incoherent background, constant coherent background, and t-averaged constant coherent background. The last method yields the best fit with the lowest χ^2/dof. The solution and method average for the σ mass and width from this best fit are

$$m_\sigma = 753 \pm 19 \text{ MeV} \quad , \quad \Gamma_\sigma = 108 \pm 53 \text{ MeV} \tag{11.1}$$

We also performed the conventional fits to spin-averaged S-wave intensity I_S. We found again that the inclusion of background (incoherent in this case) reduces the fitted value of the σ width and improves χ^2/dof. Nevertheless, the direct fits to $|\overline{S}|^2\Sigma$ are preferable at 17.2 GeV. Due to lower statistics at 5.98 and 11.85 GeV/c, we must use results for I_S to obtain σ resonance parameters. All four solutions resonate at these larger momentum transfers but yield a broader σ width: $\Gamma_\sigma = 195 \pm 81$ MeV at 5.98 GeV/c and $\Gamma_\sigma = 166 \pm 54$ MeV at 11.85 GeV/c.

We conclude that the best overall estimate of the mass and width of $\sigma(750)$ are the values in (11.1) from the best fit to $|\overline{S}|^2\Sigma$ (Table 4).

We have also examined the interference of $\sigma(750)$ with $f_0(980)$ and found that it has only a small effect on the mass and width of $\sigma(750)$. A fit to amplitude $|\overline{S}|^2\Sigma$ in the mass range above 1120 MeV shows evidence for a scalar state with average mass 1280 ± 12 MeV and width 192 ± 26 MeV.

The conventional S-wave phase shifts δ_0^0 show no evidence for the narrow $\sigma(750)$ state. It must be reiterated, that the past determinations of $\pi\pi$ phase shifts from unpolarized data on $\pi^- p \to \pi^-\pi^+ n$ assumed the absence of A_1-exchange amplitudes. This assumption is invalidated by measurements of $\pi^- p \to \pi^-\pi^+ n$, $\pi^+ n \to \pi^+\pi^- p$ and $K^+ n \to K^+\pi^- p$ on polarized targets which find large and nontrivial A_1-exchange contributions. New determinations of $\pi\pi$ phase shifts are required that do take into account the existence of A_1-exchange. Since A_1-exchange contributions are large, the revisions of $\pi\pi$ phase shifts will be significant and should provide evidence for a narrow $\sigma(750)$ state in agreement with the CERN data on polarized targets.

The mass of $\sigma(750)$ is too low for it to be a $q\overline{q}$ state. We proposed to identify $\sigma(750)$ with the lowest mass scalar gluonium $0^{++}(gg)$. This proposal is supported by the perfect agreement with the Ellis-Lanik theorem (9.1) relating the decay width of scalar gluonium $\Gamma(\sigma \to \pi^+\pi^-)$ to its mass m_σ. Another experimental support for the gluonium interpretation of $\sigma(750)$ is its absence in $\gamma\gamma \to \pi^+\pi^-$ reaction.

However, the low mass of $\sigma(750)$ is at variance with the more recent calculations of lattice QCD which predict masses of scalar gluonium above 1500 MeV.

Experiments with polarized targets have opened a whole new approach to experimental hadron spectroscopy by making accessible the study of hadron production on the level of production spin amplitudes. We may expect that this new field of amplitude spectroscopy will be further developed at the new proposed advanced hadron facilities.[83−88]

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References.

1. K.R. Popper, the Logic of Scientific Discovery, New York, Basic Books 1959 and Harper Torchbooks, 1968.
2. A. de Lesquen *et al.*, Phys. Lett. 40B, 277 (1972).
3. G. Cozzika *et al.*, Phys. Lett. 40B, 281 (1972).
4. M. Fujisaki *et al.*, Nucl. Phys. B152, 232 (1979).
5. A. Yokosawa, Phys. Rep. 64, 50 (1980).
6. I.P. Aur *et al.*, Phys. Rev. D32, 1609 (1985).
7. A. Rahbar *et al.*, Phys. Rev. Lett. 47, 1811 (1981).
8. Spin Observables of Nuclear Probes, Telluride 1988, Editor Ch. J. Horowitz.
9. M. Svec, Phys. Rev. D22, 70 (1980).
10. G. Lutz and K. Rybicki, Max Planck Institute, Munich, Internal Report No. MPI-PAE/Exp. E1.75, 1978 (unpublished).
11. G. Grayer *et al.*, Nucl. Phys. B75, 189 (1974).
12. J.G.H. de Groot, PhD Thesis, University of Amsterdam, 1978 (unpublished).
13. H. Becker *et al.*, Nucl. Phys. B150, 301 (1979).
14. H. Becker *et al.*, Nucl. Phys. B151, 46 (1979).
15. V. Chabaud *et al.*, Nucl. Phys. B223, 1 (1983).
16. K. Rybicki, I. Sakrejda, Z. Phys. C28, 65 (1985).
17. A. de Lesquen *et al.*, Phys. Rev. D32, 21 (1985).
18. A. de Lesquen *et al.*, Phys. Rev. D39, 21 (1989).
19. M. Svec, A. de Lesquen, L. van Rossum, Phys. Rev. D42 934 (1990).
20. M. Svec, A. de Lesquen, L. van Rossum, Phys. Rev. D45 55 (1992).
21. M. Svec, A. de Lesquen, L. van Rossum, Phys. Rev. D45 1518 (1992).
22. M. Svec, A. de Lesquen, L. van Rossum, Phys. Rev. D46 946 (1992).
23. M. Svec, Phys. Rev. D53, 2343 (1996).
24. S.D. Protopopescu *et al.*, Phys. Rev. D7, 1279 (1973).
25. W. Ochs, PhD Thesis, Ludwig-Maximilians-Universität, München, 1973.
26. B. Hyams *et al.*, Nucl. Phys. B64, 134 (1973).

27. P. Estabrooks, A.D. Martin, $\pi\pi$ Scattering-1973, Proceedings of the Int. Conf. on $\pi\pi$ Scattering, Tallahassee, 1973, edited by D.K. Williams and V. Hagopian, AIP Conf. Proc. No. 13 (AIP, New York, 1973), p. 37.

28. P. Estabrooks and A.D. Martin, Nucl. Phys. B79, 301 (1974).

29. P. Estabrooks and A.D. Martin, Nucl. Phys. B95, 322 (1975).

30. K.L. Au, D. Morgan, M.R. Pennington, Phys. Rev. D35, 1633 (1987).

31. B.S. Zou and D.V. Bugg, Phys. Rev. D48, R3948 (1993).

32. M. Anselmino, A. Efremov, E. Leader, Phys. Rep. 261, 1 (1995).

33. J.T. Donohue and Y. Leroyer, Nucl. Phys. B158, 123 (1979).

34. V.P. Kanavets et al., in Workshop on High Energy Spin Physics, Protvino 1995, Ed. S.B. Nurushev.

35. C.D. Lac *et al.*, J. Phys. (France) 51, 2689 (1990).

36. P.W. Johnson, R.C. Miller and G.M. Thomas, Phys. Rev. D7, 1895 (1977).

37. H. Palka, Institute of Nuclear Physics, Cracow, Internal Report No. 1230/PH, 1983, Appendix D (unpublished).

38. F. James, in Techniques and Concepts of High Energy Physics II, edited by Thomas Ferbel (Plenum, New York, 1983), p. 196.

39. J. Pišút and M. Roos, Nucl. Phys. B6 (1968) 325.

40. M. Aguilar-Bemítez and J.A. Rubio, Hadronic Resonances Part II: Experimental Analyses, published by Grupo Inter-universitario de Fisica Teorica de Atlas Energias as Report GIFT4/75. Vol. 2, 1975, Facultad de Ciencias, Universidad de Zaragoza, Spain.

41. H. Pilkuhn, The Interactions of Hadrons, North-Holland Publishing Company, 1967.

42. E Byckling and K. Kajantie, Particle Kinematics, Wiley, 1973.

43. S. Humble, Introduction to Particle Production in Hadron Physics, Academic Press 1974.

44. B.R. Martin, D. Morgan, G. Shaw, Pion-Pion Interactions in Particle Physics, Academic Press 1976.

45. I. Silin, computer code FUMILI, Long Write-up D510, CERN Computer Centre Program Library, revised 1985.

46. J.M. Blatt and V.F. Weiskopf, Theoretical Nuclear Physics, Wiley, New York, 1952.

47. N.A. Törnqvist, private e-mail correspondence.

48. W.D. Apel et al., Phys. Lett. 41B, 542 (1972).

49. D. Morgan, in Hadron 93, Como, Italy, 1993, Nuov. Cim. 107A, 1883 (1994).

50. Review of Particle Properties, Phys. Rev. D50, 1478 (1994).

51. M.R. Pennington, S.D. Protopopescu, Phys. Rev. D7, 2591 (1973).

52. M.R. Pennington, in Hadron 95, Manchester 1995, Editors M.K. Birke, G. Lafferty, J. McGovern, World Scientific 1996, p. 3.

53. T. Akesson et al., Phys. Lett. 133B, 263 (1983).

54. DM2 Collaboration, J. Augustin et al., Nucl. Phys. B320, 1 (1989).

55. N.N. Achasov, G.N. Shestakov, Phys. Rev. D49, 5779 (1994).

56. V.V. Anisovich et al., Phys. Lett. B355, 363 (1995).

57. N.A. Törnqvist, M. Roos, Phys. Rev. Lett. 76, 1575 (1996).

58. R.J. Jaffe, Phys. Rev. D15, 267 (1977).

59. R.P. Bickerstaff, B.H. McKellar, Z. Phys. C16, 171 (1982).

60. J. Weinstein and N. Isgur, Phys. Rev. D27, 588 (1983).

61. B.A. Liu and K.F. Liu, Phys. Rev. D30, 613 91984).

62. F.E. Close, Nucl. Phys. A416, 55c (1984).

63. J. Ellis and J. Lanik, Phys. Lett. 150B, 289 (1985).

64. M.S. Shifman, A.I. Vainshtein, V.I. Zakharov, Nucl. Phys. B147, 385 (1975); ibid B147, 448 (1979).

65. J.S. Bell and R.A. Bertlmann, Nucl. Phys. B177, 218 (1981); ibid B187, 285 (1981).

66. A. Bradley, C.S. Largensiepen and G. Shaw, Phys. Lett. 102B, 359 (1981).

67. Ch. Berger et al., Z. Phys. C26, 199 (1984).

68. H. Aihara et al., Phys. Rev. Lett. 57, 404 (1986).

69. A. Coureau et al., Nucl. Phys. B271, 1 (1986).

70. Z. Ajaltouni et al., Phys. Lett. 194B, 573 (1987); 197B, 565E (1987).

71. H. Marsiske et al., Phys. Rev. D41, 3324 (1990).

72. K. Ishikawa et al., Phys. Lett. 116B, 429 (1982).

73. K. Ishikawa et al., Z. Phys. C 21, 327 (1983); 21, 167 (1983).

74. B. Berg and A. Billoire, Nucl. Phys. B221, 109 (1983).

75. W.H. Hamber and M. Urs Meiller, Phys. Rev. D29, 928 (1984).

76. G. Bali et al., Phys. Lett. B309, 378 (1993).

77. F. Butler et al., Phys. Rev. Lett. 70, 2849 (1993).

78. M. Chen et al., Nucl. Phys. (Suppl.) 34, 357 (1994).

79. J.W. Dash and H. Navelet, Phys. Rev. D13, 1940 (1976).

80. G. Girardi and H. Navelet, Phys. Rev. D14, 280 (1976).

81. M. Svec, A. de Lesquen, L. van Rossum, Phys. Rev. D45, 1518 (1992).

82. M. Svec, A. de Lesquen, L. van Rossum, Phys. Lett. 220B, 653 (1989).

83. Canadian KAON Factory (TRIUMF, Vancouver, 1985).

84. Physics and a Plan for a 45 GeV Facility [LANL Report No. LA–10720–MS, Los Alamos 1986 (unpublished)].

85. Proceedings of the International Conference on a European Hadron Facility, Mainz, Germany, 1986, edited by T. Walcher [Nucl. Phys. B279, 2 (1987)].

86. M. Svec, in Proceedings of the 8th International Symposium on High-Energy Spin Physics, Minneapolis, 1988, edited by K.J. Heller, AIP Conf. Proc. No. 187 (AIP, New York, 1989), Vol. 2, p. 1181.

87. J.R. Comfort, in Proceedings of a Workshop on Science at the KAON Factory, TRIUMF, 1990, edited by D.R. Gill (unpublished), Vol. 2.

88. M. Svec, in Proceedings of a Workshop on Future Directions in Particle and Nuclear Physics at Multi-GeV Hadron Beam Facilities, BNL, 1993, edited by D.F. Geesaman (BNL Report No. BNL–52389), p. 401.

$\lvert\overline{S}\rvert^2\Sigma$ Solution	m_σ (MeV)	Γ_σ (MeV)	N_σ	χ^2/dof
Pišút-Roos shape formula				
1(MC)	736 ± 6	230 ± 32	1.40 ± 0.12	0.388
2(MC)	745 ± 12	240 ± 59	1.71 ± 0.23	0.276
1(χ^2)	738 ± 4	191 ± 16	1.50 ± 0.11	0.662
2(χ^2)	752 ± 10	253 ± 46	1.79 ± 0.15	0.968
Phenomenological shape formula				
1(MC)	732 ± 6	231 ± 33	6.50 ± 0.57	0.418
2(MC)	740 ± 11	241 ± 61	7.94 ± 1.11	0.288
1(χ^2)	733 ± 4	192 ± 16	7.00 ± 0.50	0.740
2(χ^2)	747 ± 10	256 ± 47	8.29 ± 0.74	0.986

Table 1. Results of the fits to the mass distribution $|\overline{S}|^2\Sigma$ measured in $\pi^-p \to \pi^-\pi^+n$ at 17.2 GeV/c using a single Breit-Wigner formula (4.1). The notation MC and χ^2 indicates the solutions obtained by the Monte Carlo and χ^2 minimization methods, respectively.

$\|\overline{S}\|^2\Sigma$ Solution	m_σ (MeV)	Γ_σ (MeV)	B	N_S	χ^2/dof
Phenomenological shape formula					
1(MC)	731 ± 6	202 ± 110	0.15 ± 0.55	5.75 ± 2.54	0.416
2(MC)	744±14	103 ± 74	0.73 ± 0.49	5.11 ± 1.84	0.144
1(χ^2)	736 ± 4	147 ± 43	0.19 ± 0.17	6.13 ± 0.84	0.696
2(χ^2)	745 ± 41	98 ± 41	0.70 ± 0.28	5.70 ± 1.30	0.626

Table 2. Results of the fits to the mass distribution $|\overline{S}|^2\Sigma$ measured in $\pi^- p \to \pi^-\pi^+ n$ at 17.2 GeV/c using a Breit-Wigner formula with a constant incoherent background (4.6). The notation MC and χ^2 as in Table 1.

$\lvert\overline{S}\rvert^2\Sigma$ Solution	m_σ (MeV)	Γ_σ (MeV)	B_1	B_2	N_S	χ^2/dof
Phenomenological shape formula						
1(MC)	770 ± 19	114 ± 17	0.90 ± 0.43	1.34 ± 0.84	1.09 ± 0.61	0.136
2(MC)	745 ± 31	104 ± 76	0.02 ± 1.07	1.84 ± 1.14	1.09 ± 0.90	0.144
$1(\chi^2)$	761 ± 13	138 ± 19	0.34 ± 0.16	$0.69 \pm .045$	2.41 ± 1.14	0.362
$2(\chi^2)$	738 ± 20	103 ± 112	-0.17 ± 0.65	1.30 ± 0.50	1.79 ± 0.86	0.898

Table 3. Results of the fits to the mass distribution $\lvert\overline{S}\rvert^2\Sigma$ measured in $\pi^- p \to \pi^-\pi^+ n$ at 17.2 GeV/c using a Breit-Wigner formula with constant coherent background (4.15). The notation MC and χ^2 as in Table 1.

$\lvert\overline{S}\rvert^2\Sigma$ Solution	m_σ (MeV)	Γ_σ (MeV)	B	B	N_S	χ^2/dof
Phenomenological shape formula						
1(MC)	774 ± 14	101 ± 44	0.73 ± 0.31	0.29 ± 0.14	3.99 ± 0.96	0.108
2(MC)	744 ± 31	103 ± 79	0.73 ± 0.54	0.01 ± 0.23	5.10 ± 1.90	0.144
1(χ^2)	761 ± 12	134 ± 41	0.25 ± 0.17	0.15 ± 0.07	5.74 ± 0.82	0.362
2(χ^2)	733 ± 20	93 ± 48	0.80 ± 0.39	-0.12 ± 0.19	5.31 ± 1.53	0.592

Table 4. Results of the fits to the mass distribution $|\overline{S}|^2\Sigma$ measured in $\pi^- p \to \pi^-\pi^+ n$ at 17.2 GeV/c using a Breit-Wigner formula with t-averaged constant coherent background (4.17). The notation MC and χ^2 as in Table 1.

$\lvert\overline{S}\rvert^2\Sigma$ Solution	m_σ (MeV)	Γ_σ (MeV)	B_1	B_2	N_S	χ^2/dof
Phenomenological shape formula						
1(MC, χ^2)	778 ± 13	95 ± 27	1.20 ± 0.35	0.85 ± 0.33	1.24 ± 0.39	0.096
2(MC, χ^2)	758 ± 32	135 ± 49	0.54 ± 0.69	1.00 ± 0.28	1.85 ± 0.83	0.162
			C_1	C_2		
1(MC, χ^2)			0.42 ± 0.52	1.25 ± 0.39		
2(MC, χ^2)			-0.35 ± 0.67	0.97 ± 0.55		

Table 5. Results of the fit to the mass distribution $\lvert\overline{S}\rvert^2\Sigma$ in the mass range from 600 to 1120 MeV taking into account the interference of $\sigma(750)$ with $f_0(980)$ using the parametrization (5.3). The notation MC and χ^2 as in Table 1.

$\lvert\overline{S}\rvert^2\Sigma$ Solution	m (MeV)	Γ (MeV)	B	N	χ^2/dof
1(χ^2)	1284 ± 12	209 ± 29	0.001 ± 0.32	5.96 ± 0.62	1.393
2(χ^2)	1276 ± 11	175 ± 24	0.001 ± 0.09	6.21 ± 0.70	1.738

Table 6. The results of the fit to the mass distribution $\lvert\overline{S}\rvert^2\Sigma$ in the $f_0(1300)$ mass region from 1120 to 1520 MeV using a single Breit-Wigner formula with incoherent constant background. The notation χ^2 as in Table 1.

I_S Solution	m_σ (MeV)	Γ_σ (MeV)	B	N_S	χ^2/dof
Single Breit-Wigner fit					
(1,1)	766 ± 5	258 ± 19	–	1.98 ± 0.07	0.450
(1,2)	769 ± 12	263 ± 45	–	2.26 ± 0.17	0.498
(2,1)	766 ± 10	255 ± 37	–	2.19 ± 0.15	0.240
(2,2)	768 ± 12	303 ± 49	–	2.48 ± 0.16	0.816
Breit-Wigner fit with constant background					
(1,1)	767±5	210 ± 43	0.19 ± 0.17	1.69 ± 0.23	0.365
(1,2)	768 ± 12	209 ± 99	0.20 ± 40	1.92 ± 0.61	0.470
(2,1)	766 ± 9	208 ± 82	0.17 ± 0.32	1.91 ± 0.47	0.218
(2,2)	765 ± 10	188 ± 76	0.41 ± 0.34	1.85 ± 0.42	0.700

Table 7. Results of the fits to the four solutions of the S-wave intensity measured in $\pi^- p \to \pi^- \pi^+ n$ at 17.2 GeV/c using Monte Carlo method for amplitude analysis. The fits are made with Breit-Wigner parametrization (6.3) and (6.4) with the Pišút-Roos shape factor.

I_S Solution	m_σ (MeV)	Γ_σ (MeV)	B	N_S	χ^2/dof
Single Breit-Wigner fit					
(1,1)	760 $\pm$ 8	269 $\pm$ 29	–	2.00 $\pm$ 0.13	0.414
(2,2)	786 $\pm$ 21	408 $\pm$ 90	–	2.24 $\pm$ 0.16	1.140
Breit-Wigner fit with constant background					
(1,1)	761$\pm$8	227 $\pm$ 68	0.12 $\pm$ 0.19	1.84 $\pm$ 0.28	0.394
(2,2)	780 $\pm$ 13	187 $\pm$ 77	0.63 $\pm$ 0.31	1.51 $\pm$ 0.31	0.864

Table 8. Results of the fits to two of four solutions of the S-wave intensity measured in $\pi^- p \to \pi^- \pi^+ n$ at 17.2 GeV/c using χ^2 minimization method for amplitude analysis. The fits are made with Breit-Wigner parametrization (6.3) and (6.4) with the Pišút-Roos shape factor.

I_S Solution	m_σ (MeV)	Γ_σ (MeV)	B	N_S	χ^2/dof
Single Breit-Wigner fit					
(1,1)	723 ± 22	282 ± 68	–	0.53 ± 0.10	0.888
(1,2)	696 ± 36	333 ± 128	–	1.13 ± 0.34	0.118
(2,1)	740 ± 32	296 ± 116	–	1.02 ± 0.29	0.204
(2,2)	714 ± 27	362 ± 102	–	1.52 ± 0.30	0.194
Breit-Wigner fit with constant background					
(1,1)	746 ± 16	145 ± 69	0.18 ± 0.10	0.59 ± 0.18	0.712
(1,2)	706 ± 39	262 ± 24	0.13 ± 0.39	1.05 ± 0.44	0.114
(2,1)	745 ± 30	165 ± 112	0.23 ± 0.21	0.97 ± 0.42	0.094
(2,2)	724 ± 25	211 ± 117	0.25 ± 0.20	1.41 ± 0.42	0.124

Table 9. Results of the fits to the four solutions of the S-wave intensity measured in $\pi^+ n \to \pi^+\pi^- p$ at 5.98 GeV/c using Monte Carlo method for amplitude analysis. The fits are made with Breit-Wigner parametrization (6.3) and (6.4) with the Pišút-Roos shape factor.

I_S Solution	m_σ (MeV)	Γ_σ (MeV)	B	N_S	χ^2/dof
Single Breit-Wigner fit					
(1,1)	778 ± 10	158 ± 21	–	1.19 ± 0.12	2.158
(1,2)	749 ± 31	353 ± 88	–	1.47 ± 0.25	0.430
(2,1)	752 ± 20	237 ± 52	–	1.50 ± 0.27	0.844
(2,2)	749 ± 19	309 ± 63	–	1.92 ± 0.24	0.632
Breit-Wigner fit with constant background					
(1,1)	782±9	117 ± 26	0.08 ± 0.03	1.13 ± 0.16	1.024
(1,2)	770 ± 24	202 ± 74	0.09 ± 0.04	1.52 ± 0.32	0.080
(2,1)	763 ± 18	153 ± 55	0.12 ± 0.05	1.46 ± 0.39	0.236
(2,2)	756 ± 15	200 ± 59	0.11 ± 0.05	1.93 ± 0.33	0.212

Table 10. Results of the fits to the four solutions of the S-wave intensity measured in $\pi^+ n \to \pi^+ \pi^- p$ at 11.85 GeV/c using Monte Carlo method for amplitude analysis. The fits are made with Breit-Wigner parametrization (6.3) and (6.4) with the Pišút-Roos shape factor.

Figure Captions.

Fig. 1. Mass dependence of unnormalized amplitudes $|\overline{S}|^2\Sigma$ and $|S|^2\Sigma$ measured in $\pi^- p_\uparrow \to \pi^-\pi^+ n$ at 17.2 GeV/c at $-t = 0.005 - 0.20$ (GeV/c)^2 using the Monte Carlo method for amplitude analysis (Ref. 23). Both solutions for the amplitude $|\overline{S}|^2\Sigma$ resonate at 750 MeV while the amplitude $|S|^2\Sigma$ is nonresonating in both solutions.

Fig. 2. Mass dependence of unnormalized amplitudes $|\overline{S}|^2\Sigma$ and $|S|^2\Sigma$ measured in $\pi^- p_\uparrow \to \pi^-\pi^+ n$ at 17.2 GeV/c at $-t = 0.005 - 0.20$ (GeV/c)^2 using the χ^2 minimization method for amplitude analysis. Based on Fig. 10 of Ref. 13 and Fig. VI-21 of Ref. 12. Both solutions for the amplitude $|\overline{S}|^2\Sigma$ resonate at 750 MeV while the amplitude $|S|^2\Sigma$ is nonresonating in both solutions. The analysis used the same data as in Fig. 1 (20 MeV mass bins).

Fig. 3. The fits to amplitude $|\overline{S}|^2\Sigma$ using the single Breit-Wigner parametrization (4.1) with Pišút-Roos shape factor (4.2a). The fitted parameters are given in Table 1.

Fig. 4. The fits to amplitude $|\overline{S}|^2\Sigma$ using the single Breit-Wigner parametrization (4.1) with phenomenological shape factor $F = 1$. The fitted parameters are given in Table 1.

Fig. 5. The fits to amplitude $|\overline{S}|^2\Sigma$ using the Breit-Wigner parametrization (4.6) with constant incoherent background and with phenomenological shape factors $F = 1$. The fitted parameters are given in Table 2.

Fig. 6. The fits to amplitude $|\overline{S}|^2\Sigma$ using the Breit-Wigner parametrization (4.15) with constant coherent background and with phenomenological shape factor $F = 1$. The fitted parameters are given in Table 3.

Fig. 7. The fits to amplitude $|\overline{S}|^2\Sigma$ using the Breit-Wigner parametrization (4.19) with t-averaged constant coherent background and with phenomenological shape factor $F = 1$. The fitted parameters are given in Table 4.

Fig. 8. Mass dependence of unnormalized amplitudes $|\overline{S}|^2\Sigma$ and $|S|^2\Sigma$ measured in $\pi^- p \to \pi^-\pi^+ n$ at 17.2 GeV/c at $-t = 0.005 - 0.20$ (GeV/c)^2 in 40 MeV mass bins from 600 to 1520 MeV. Based on Figs. 2 and 6 from Ref. 14. The amplitude $|\overline{S}|^2\Sigma$ resonates at 750 in Solution 1 and at 800 MeV in Solution 2 while the amplitude for $|S|^2\Sigma$ is nonresonating in both solutions in this mass range.

Fig. 9. The fits to amplitude $|\overline{S}|^2\Sigma$ using the Breit-Wigner parametrization (5.4) below 1120 MeV and a single Breit-Wigner formula with incoherent background above 1120 MeV. The phenomenological shape factor $F = 1$. The fitted parameters are given in Tables 5 and 6.

Fig. 10. Four solutions for the S-wave intensity I_S measured in the reaction $\pi^- p_\uparrow \to \pi^-\pi^+ n$ at 17.2 GeV/c and $-t = 0.005 - 0.2$ (GeV/c)^2 using Monte Carlo method for amplitude analysis (Ref. 23). The solid curves are fits to single Breit-Wigner parametrization (6.3). The dashed curves are fits to Breit-Wigner parametrization (6.4) with incoherent background. The fitted parameters are given in Table 7.

Fig. 11. Two of the four solutions for the S-wave intensity I_S measured in the $\pi^- p_\uparrow \to \pi^-\pi^+ n$ at 17.2 GeV/c and $-t = 0.005 - 0.2$ (FeV/c)^2 using χ^2 minimization method for amplitude analysis. The data are based on Fig. 14a of Ref. 13 and Fig. 12 of Ref. 11. The solid and dashed curves are Breit-Wigner fits as in Fig. 10. The fitted parameters are given in Table 8.

Fig. 12. Four solutions for the S-wave intensity I_S measured in $\pi^+ n_\uparrow \to \pi^+\pi^- p$ at 5.98 GeV/c and $-t = 0.2 - 0.4$ (GeV/c)^2 using Monte Carlo method for amplitude analysis (Ref. 23). The solid and dashed curves are Breit-Wigner fits as in Fig. 10. The fitted parameters are given in Table 9.

Fig. 13. Four solutions for the S-wave intensity I_S measured in $\pi^+ n_\uparrow \to \pi^+\pi^- p$ at 11.85 GeV/c and $-t = 0.2 - 0.4$ (GeV/c)^2 using Monte Carlo method for amplitude analysis (Ref. 23). The solid and dashed curves are Breit-Wigner fits as in Fig. 10. The fitted parameters are given in Table 10.

Fig. 14. The t-evolution of mass dependence of moduli squared of t-channel normalized transversity amplitudes $|L|^2$, $|\overline{L}|^2$, $|U|^2$ and $|\overline{U}|^2$ in $\pi^+ n_\uparrow \to \pi^+\pi^- p$ at

5.98 GeV/c together with results for $\pi^- p_\uparrow \to \pi^-\pi^+ n$ at 17.2 GeV/c and $t = 0.068$ $(\mathrm{GeV/c})^2$.

Fig. 15. The ratio of amplitudes with recoil nucleon transversity "down" and "up" with dimeson helicity $\lambda = 0$. The deviation from unity shows the strength of A_1-exchange amplitudes. Based on Fig. 6 of Ref. 14. In our notation, $g_S = S$, $h_S = \overline{S}$, $g_P = L$, $h_P = \overline{L}$.

Fig. 16. Test of predictions $\rho^y_{ss} + \rho^y_{00} + 2\rho^y_{11} = -2(\rho^y_{00} - \rho^y_{11}) = +2\rho^y_{1-1}$ due to vanishing A_1-exchange in $\pi^- p_\uparrow \to \pi^-\pi^+ n$ at 17.2 GeV/c and $-t = 0.005 - 0.2$ $(\mathrm{GeV/c})^2$.

Fig. 17. Test of predictions $\mathrm{Re}\rho^y_{10} = \mathrm{Re}\rho^y_{1s} = \mathrm{Re}\rho^y_{0s} = 0$ due to vanishing A_1-exchange in $\pi^- p_\uparrow \to \pi^-\pi^+ n$ at 17.2 GeV/c and $-t = 0.005 - 0.2$ $(\mathrm{GeV/c})^2$.

Fig. 18. S-wave intensity normalized to 1 at maximum value. The data correspond to solutions $I_S(1,1)$ and $I_S(2,2)$ at 17.2 (GeV/c) from Ref. 23. The smooth curves are predictions of phase shift analysis for $\pi^+\pi^- \to \pi^+\pi^-$ from Ref. 27. The dashed curve is the accepted solution Down, the dot-dashed curve is the rejected solution Up.